

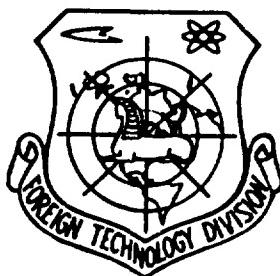
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FTD-MT-24-41-69

FOREIGN TECHNOLOGY DIVISION



CENTRAL AEROLOGICAL OBSERVATORY
(SELECTED ARTICLES)



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REF ID: A72147
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FTD-MT-24-41-69

EDITED MACHINE TRANSLATION

CENTRAL AEROLOGICAL OBSERVATORY (SELECTED ARTICLES)

English pages: 54

SOURCE: Tsentral'naya Aerologicheskaya
Observatoriya. Trudy (Transactions),
No. 78, 1967, pp. 57-62, 80-99,
100-108.

UR/2789-67-000-078

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FOREIGN TECHNOLOGY DIVISION
AF-APB, OHIO

FTD-MT-24-41-69

Date 22 May 1969

DATA HANDLING PAGE					
01-ACCESSION NO. 90-DOCUMENT LOC TT9500774		39-TOPIC TAGS atmospheric turbulence, atmospheric stratification, wave number, atmospheric disturbance			
09-TITLE TURBULENCE SPECTRUM OF A STABLY STRATIFIED ATMOSPHERE					
47-SUBJECT AREA 04					
12-AUTHOR/CO-AUTHORS SHUR, G. N.				10-DATE OF INFO -----67	
43-SOURCE TSENTRAL'NAYA AEROLOGICHESKAYA OBSERVATORIYA. TRUDY (RUSSIAN)				FTD-	68-DOCUMENT NO. MT-24-41-69
					69-PROJECT NO. 72302-78
63-SECURITY AND DOWNGRADING INFORMATION UNCL, O			64-CONTROL MARKINGS NONE	97-HEADER CLASH UNCL	
76-REEL FRAME NO. 1889 0103	77-SUPERSEDES	78-CHANGES	40-GEOGRAPHICAL AREA UR	NO OF PAGES 14	
CONTRACT NO.	X REF ACC. NO. 65-AT7034014	PUBLISHING DATE 94-00	TYPE PRODUCT TRANSLATION	REVISION FREQ NONE	
STEP NO. 02-UR/2789/67/000/078/0100/0108			ACCESSION NO.		
ABSTRACT <p>(U) The author discusses the turbulence spectrum governing the spectral density of energy distribution of turbulence motion or of the energy spectra of turbulence as formulated by Obukhov-Kolmogorov. He also reviews various experimental studies confirming the presence of buoyancy forces in the spectrum of the turbulence interval of wave numbers. Experimental curves and equations are presented showing that buoyancy subregions exist in the spectrum of developed continuous turbulence during stable stratification of the atmosphere. In the case of stable wave disturbances the spatial energy flux retains its direction towards large wave numbers. The energy arriving in large scales is transmitted to the region of submolecular scales. In this case a discrete spectrum is obtained. If the medium possesses viscosity a "pumping out" of energy occurs from any mechanical motion forming in this region. In the case of very strong thermal stability of the medium, the spectrum of turbulence may be localized in a wide zone of wave numbers. In the case of less thermal atmospheric stability, as in the case of large dynamic instability, and also in the region of wave numbers far from the subregion of buoyancy, the turbulence spectrum obtained is considerably wider and differs substantially from spectra of purely harmonic fluctuations. The most frequent case is that of a continuous spectrum of turbulence in which kinetic energy is transferred from small to large scales. Orig. art. has: 5 figures, 11 formulas.</p>					

DATA HANDLING PAGE

01-ACCESSION NO. 98-DOCUMENT LOC		39-TOPIC TAGS		
TT9500772		atmospheric turbulence, atmospheric stratification, aeronautic meteorology, wind velocity, atmospheric sounding		
05-TITLE DAILY VARIATION IN THE TURBULENCE COEFFICIENT ABOVE FLAT TERRAIN				
07-SUBJECT AREA				
04				
12-AUTHOR/CO-AUTHORS		10-DATE OF INFO -----67		
LITVINNOVA, V. D.		13-SOURCE TSENTRAL'NAYA AEROLOGICHESKAYA OBSERVATORIYA. TRUDY (RUSSIAN)		
		FTD	68-DOCUMENT NO. MT-24-41-69	
			69-PROJECT NO. 72302-78	
63-SECURITY AND DOWNGRADING INFORMATION		64-CONTROL MARKINGS	97-HEADER CLASH	
UNCL, O		NONE	UNCL	
76-REEL FRAME NO.	77-SUPERSEDES	78-CHANGES	80-GEOGRAPHICAL AREA	NO OF PAGES
1889 0101			UR	9
CONTRACT NO.	X REF ACC. NO.	PUBLISHING DATE	TYPE PRODUCT	REVISION FREQ
	65-AT/034011	94-00	TRANSLATION	NONE
STEP NO.		ACCESSION NO.		
02-UR/2789/07/000/078/0057/0062				

ABSTRACT

(U) The results of aircraft investigations of atmospheric turbulence between 100 m to 3 km in the area around the city of Dolgoprudnyy carried out by the Central Aerological Observatory in 1946 are analyzed. A meteorograph and an accelerograph for recording acceleration in a vertical plane were installed on the aircraft. During the flights vertical soundings were made from areas at heights of 100, 300, 500, 1000, 1500, 2000, and 3000 m and at the surface of the earth. The turbulent exchange (K) was computed. Individual flights analyzed with the aid of graphs of the vertical profile of the turbulence coefficient and the daily variation of the turbulence coefficient at different altitudes. Under anticyclonic conditions the turbulence coefficient has a distinct vertical profile and its maximum is around 13 hours at all levels. Frequently one to two maxima were observed and their values are determined by the thermal stratification of the atmosphere. The turbulence coefficient manifests a daily variation. The maximum K values are reached simultaneously at all levels in the daytime, but the beginning of daily variations is most clearly manifested at 100 m. Grig. art. has: 3 figures.

DATA HANDLING PAGE

01-ACCESSION NO. 03-DOCUMENT LOC TT9500773		02-TOPIC TAGS atmospheric turbulence, turbulent flow, atmospheric wind field		
05-TITLE THEORY OF SPATIAL- TEMPORAL CORRELATION OF VELOCITIES IN AN ISOTROPIC TURBULENT FLOW				
07-SUBJECT AREA 04				
12-AUTHORS/CO-AUTHORS SMIRNOV, V. I.; 16-SHAPIRO, B. SH.		10-DATE OF INFO -----67		
13-SOURCE TSENTRAL'NAYA AEROLOGICHESKAYA OBSERVATORIYA. TRUDY (RUSSIAN)		14-DOCUMENT NO. FTD-MT-24-41-69 15-PROJECT NO. 72302-78		
18-SECURITY AND DOWNGRADING INFORMATION UNCL, O		64-CONTROL MARKINGS NONE		97-HEADER CLASH UNCL
76-REEL FRAME NO. 1889 0102	77-SUPERSEDES CONTRACT NO.	78-CHANGES X REF ACC. NO. 55-AT7034013	40-GEOGRAPHICAL AREA UR	NO OF PAGES 31
		PUBLISHING DATE 94-00	TYPE PRODUCT TRANSLATION	REVISION FREQ NONE
STEP NO. 02-UR/2789/67/000/078/0080/0099		ACCESSION NO.		
<p>ABSTRACT</p> <p>(U) The deductive turbulence theory proposed by Chandrasekhar is based on postulates of homogeneity, isotropy and stationary state and also on the hypothesis of quasi-normality (hypothesis of Millionshchikov) associating correlation tensors of the second and fourth rank. In this paper the authors generalize the equations of Chandrasekhar for the case of non-stationary turbulence. There is derived a dynamic equation for determining the scalar correlation tensors describing a two point space-time relation of the velocities of turbulent flow and constituting a generalization of the equation of Chandrasekhar for stationary turbulence. In the derivation there were assumed conditions of homogeneity and isotropy and also the generalized hypothesis of quasi-normal distribution of velocities. In the case of the direct derivation of the stationary equation there is obtained a pair of equations that have not been considered by Chandrasekhar. But these equations have to be disregarded because they cannot be obtained from a non-stationary equation. In principle it is possible to obtain another equation for non-stationary turbulence which has not yet been investigated. The generalized equation of Chandrasekhar correctly describes the last stage of the origin of turbulence. The stationary equation of Chandrasekhar for the spatial-temporal longitudinal correlation function can be solved by representing the solution in a power series. Orig. art. has: 2 figures, 112 formulas.</p>				

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	А а	A, a	Р р	Р р	R, r
Б б	Б б	B, b	С с	С с	S, s
В в	В в	V, v	Т т	Т т	T, t
Г г	Г г	G, g	Ү ү	Ү ү	U, u
Д д	Д д	D, d	Ф ф	Ф ф	F, f
Е е	Е е	Ye, ye; E, e*	Х х	Х х	Kh, kh
Ж ж	Ж ж	Zh, zh	Ц ц	Ц ц	Ts, ts
З з	З з	Z, z	Ч ч	Ч ч	Ch, ch
И и	И и	I, i	Ш ш	Ш ш	Sh, sh
Й я	Й я	Y, y	Щ щ	Щ щ	Shch, shch
К к	К к	K, k	Ђ ђ	Ђ ђ	"
Л л	Л л	L, l	Ң ң	Ң ң	Y, y
М м	М м	M, m	Ӯ Ӯ	Ӯ Ӯ	'
Н н	Н н	N, n	Э э	Э э	E, e
О о	О о	O, o	Ӯ Ӯ	Ӯ Ӯ	Yu, yu
П п	П п	P, p	Я я	Я я	Ya, ya

* ye initially, after vowels, and after Յ, Յ; ē elsewhere.
 When written as ё in Russian, transliterate as yё or ё.
 The use of diacritical marks is preferred, but such marks
 may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH
DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csech
arc sin	\sin^{-1}
arc cos	\cos^{-1}
arc tg	\tan^{-1}
arc ctg	\cot^{-1}
arc sec	\sec^{-1}
arc cosec	\csc^{-1}
arc sh	\sinh^{-1}
arc ch	\cosh^{-1}
arc th	\tanh^{-1}
arc cth	\coth^{-1}
arc sch	\sech^{-1}
arc csch	\csech^{-1}
rot	curl
lg	log

DAILY VARIATION IN THE TURBULENCE COEFFICIENT ABOVE FLAT TERRAIN

V. D. Litvinova

Results are given of aircraft investigations of turbulence around Dolgoprudnyy from an altitude of 100 m to 3 km. It was found that under anti-cyclonic conditions the maximum value of the turbulence coefficient at all levels is observed around 1300 hours. The value of the maximum depends on the thermal stratification of atmosphere. Diurnal variations in the turbulence coefficient are most noticeable at a height of 100 m.

For investigation of a number of physical processes in the atmosphere it is necessary to know the intensity of turbulent exchange of heat, moisture, and momentum.

Experimental investigations of the conditions of appearance and distribution of turbulence were conducted at TsAO (Central Aerological Observatory), GGO (Main Geophysical Observatory), and others. For free atmosphere the most detailed investigations of variability of the turbulence coefficient with time, height, and horizontal extent are given in the works of P. A. Vorontsov [1], N. Z. Pinus [5], and V. Ye. Minervin [4].

In this work, according to data from flights conducted by TsAO in 1946 on a PO-2 aircraft in the Dolgoprudnyy area, certain characteristics of turbulent exchange in the lower troposphere are examined.

The aircraft carried a meteorograph and an accelerograph of type SP-11, intended for recording g of the aircraft in the vertical plane. During flights vertical sounding was made of the atmosphere with platforms at heights of 100, 300, 500, 1000, 1500, 2000, and 3000 m above the earth's surface. The duration of recording of accelerograph on these platforms did not exceed one to two minutes. In all 48 flights were made, one or two flights a day, with four series of quickened soundings of the atmosphere being made on three to five flights daily.

In 1965 all data from these flights was reanalyzed, coefficients of turbulent exchange were calculated, and graphs of dependence of the coefficient of exchange on heights and time of day were plotted. The coefficient of turbulent exchange was calculated by the Iyapin-Dubov formula, definitized to M. A. German in work [2]:

$$K = \frac{\bar{V}|\bar{W}| \Delta t}{2\eta}$$

where \bar{V} - average flight speed of aircraft on given platform, $|\bar{W}|$ - average absolute velocity of vertical gust of air, during the calculation of which the mechanical and flight characteristics of the aircraft were considered, Δt - average time of preservation of sign of pulsations in speed, η - value of transfer function for the given type of aircraft, given in work [3].

Below will be examined only cases of quickened sounding of the atmosphere.

1. Flights of 11 August 1946

On 11 August 1946 three flights were made at 13:17, 15:52, and 18:30 hours. Flights passed through the central part of an anticyclone. Variable overcast with weak north and northwest winds was observed. Platforms were at heights of 100, 500, 1000, 1500, and 3000 m.

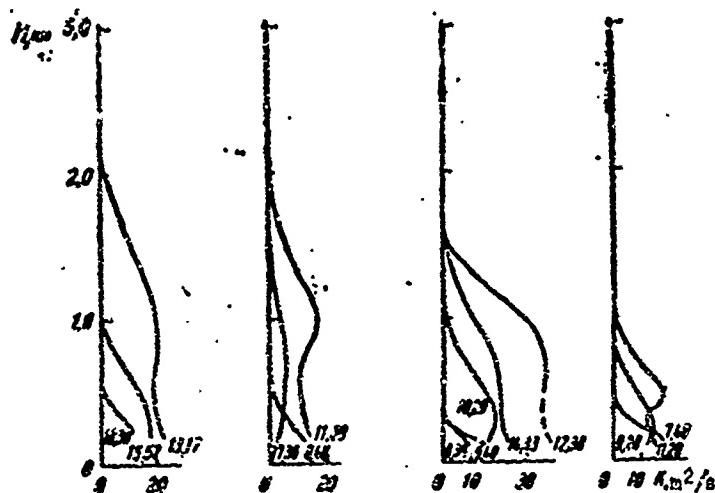


Fig. 1. Vertical profiles of the turbulence coefficient. a) 11 Aug 1946,
b) 12 Aug 1946, c) 29 Sep 1946,
d) 25 Oct 1946.

As can be seen from Fig. 1a during the flight of the aircraft at 1317 hours the value of K at a height of 100 m was $24 \text{ m}^2/\text{s}$, at 500 m - $18 \text{ m}^2/\text{s}$, and at 1000 m - $21 \text{ m}^2/\text{s}$. The value of K then decreases rapidly to $2 \text{ m}^2/\text{s}$ at a height of 2 km. At a height of 3 km the flight was calm. The vertical temperature gradient varied from 0.7 to $0.9^\circ/100 \text{ m}$. At a height of 2.8 km there was an inversion. At a height of 1.5 km was the lower FrCu and Cu cong boundary. Unfortunately, no g recording was made at this height.

In 1552 hours the value of the turbulence coefficient at 100 m was $18 \text{ m}^2/\text{s}$, at 500 m - $15 \text{ m}^2/\text{s}$, and from 1000 m the flight was calm. The vertical temperature gradient to a height of 660 m was $1^\circ/100 \text{ m}$, and at higher altitudes $0.75^\circ/100 \text{ m}$. At 2.8 km $\gamma = 0.1/100 \text{ m}$.

At 1830 hours the turbulence coefficient at 100 m was $12 \text{ m}^2/\text{s}$, and at 500 m the flight was calm. The value of γ changed from 0.75 to $1^\circ/100 \text{ m}$, and at 2.8 km a temperature inversion was observed.

As can be seen from Fig. 1a, maximum intensity of turbulence was observed at 1300 hours, and it gradually weakened toward 1800; at a height of 100 m the value of the turbulence coefficient decreased from $24 \text{ m}^2/\text{s}$ (at 1300) to $12 \text{ m}^2/\text{s}$ (at 1800), and at

500 m decay of turbulence toward evening was still faster: from $23 \text{ m}^2/\text{s}$ at 1300 to $15 \text{ m}^2/\text{s}$ at 1500, and at 1800 at this height flight was absolutely calm. As can be seen, changes of turbulence coefficient with height follow well the changes with height of the vertical temperature gradient, especially in daytime hours. With decrease of the vertical temperature gradient of the value of K decreases.

2. Flights of 12 August 1946

Flights passed through the central part of an anticyclone during scantily clouded weather with weak northwest winds at the surface. Flights were made at 0552, 0848, 1139, and 1736 hours at heights of 100, 500, 1000, 2000, and 3000 m. The turbulence coefficient was calculated for only the last three flights, since the first was calm. At 100 m at 0848 hours the value of $K = 16 \text{ m}^2/\text{s}$ (Fig. 1b). In the layer to 300 m were observed raised values of vertical temperature and wind gradients ($\gamma = 0.92^\circ/100 \text{ m}$, $\beta = 0.83 \text{ m/s}/100 \text{ m}$). In the 300-2560 m layer values of γ did not exceed $0.85^\circ/100 \text{ m}$, and β was not over $0.35 \text{ m/s}/100 \text{ m}$. At 2560 m was an inversion.

Further flight in the entire layer sounded was calm. In 1139 hours the on vertical profile of the turbulence coefficient two maxima are observed. The first maximum, caused by the influence of the underlying surface, is at a height of 100 m; the value of K here is $16 \text{ m}^2/\text{s}$. The value of the vertical temperature gradient is $0.86^\circ/100 \text{ m}$, and that of wind is $7.1 \text{ m/s}/100 \text{ m}$. At 500 m the value of K is $11 \text{ m}^2/\text{s}$, and then again increases, attaining $18 \text{ m}^2/\text{s}$ at a height of 100 m. Appearance of the second maximum of the turbulence coefficient can be explained, apparently, by thermal stratification of the atmosphere. For this layer there characteristically is presence of a large value of vertical temperature gradient ($\gamma = 0.97^\circ/100 \text{ m}$) and sharp decrease of vertical wind gradient (to $0.35 \text{ m/s}/100 \text{ m}$). Above this level flight was calm. By 1736 hours there is decay of turbulence in the entire sounded layer. Values of K at all heights to 1000 m by this time no longer exceeded

several m/s. At heights of 2000 and 3000 m flight was calm. The vertical temperature gradient in the 300-1300 m layer equaled $0.82-0.94^{\circ}/100$ m and at the surface of earth and at high altitudes was $0.64-0.73^{\circ}/100$ m. The value of β to a height of 3000 m was 2.72 m/s/100 m, and further 0.1-0.3 m/s/100 m.

3. Flights of 29 September 1946

Flights passed through the spur of an anticyclone. Cloudless weather with weak north winds at the surface was observed.

Data obtained in this day are the most interesting. During the day six flights were made. Therefore it was possible to trace most fully the atmospheric composition to a height of 2 km. Flights were made at 0640, 0838, 1029, 1238, 1433, and 1625 hours at heights of 100, 300, 500, 1000, 1500, and 2000 m.

According to data from five flights vertical profiles of the turbulence coefficient were plotted. For the 1625 flight the turbulence coefficient was not calculated, since turbulence was not observed continuously, but only at times.

As can be seen from Fig. 1c, at 0640 the turbulence coefficient at 100 m was $17 \text{ m}^2/\text{s}$, while at higher altitudes flight was calm. In the layer of atmosphere from the earth to 400 m a deep inversion was observed. By 0838 the temperature inversion had vanished and the vertical temperature gradient increased rapidly to $1.00^{\circ}/100$ m, while the wind gradient increased to 3.75 m/s/100 m. At 100 m $K = 12 \text{ m}^2/\text{s}$. By 1029 the vertical profile of K significantly changes. On the figure is seen a well-defined maximum of the turbulence coefficient in the layer from the earth to 500 m, where $K = 14-19 \text{ m}^2/\text{s}$. At heights from 1000 m to 2000 m flight was calm. Values of vertical temperature gradient to a height of 1100 m varied from 0.8 to $1.0^{\circ}/100$ m, and wind gradient varied from 0.2 to 1.0 m/s/100 m. By 1238 hours atmospheric turbulence was strengthened sharply. On the profile of the turbulence coefficient

two maxima are observed. At a height of 100 m $K = 40 \text{ m}^2/\text{s}$, and then there is a certain tendency toward slight decrease at a height of 300 m, after which the value of K increases to 500 m, where it attains $35 \text{ m}^2/\text{s}$. At 1000 m $K = 31 \text{ m}^2/\text{s}$. The vertical temperature gradient in this layer is equal to $1^\circ/1000 \text{ m}$, and that of wind is $0.15-0.58 \text{ m/s}/100 \text{ m}$. At heights of 1500 and 2000 m flight was calm.

By 1433 the vertical profile of K has more smoothed form. In the layer from 100 to 500 m the turbulence coefficient was $20-24 \text{ m}^2/\text{s}$ and at 1000 m was $10 \text{ m}^2/\text{s}$. In this whole layer the vertical temperature gradient is equal to $1^\circ/100 \text{ m}$, that of wind to $0.15-1.00 \text{ m/s}/100 \text{ m}$. At 2000 m flight was calm. By 1625 almost full decay of turbulence had taken place.

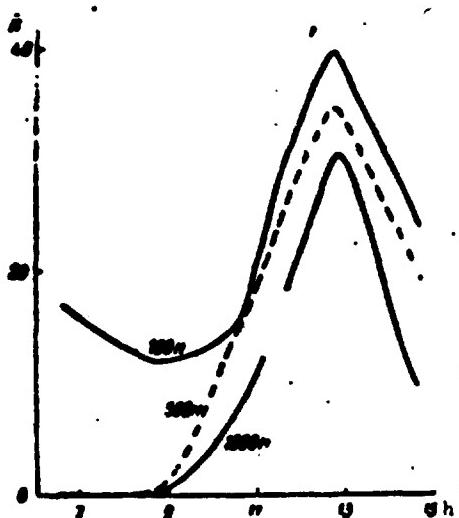


Fig. 2. Daily variation of turbulence coefficient for various heights.

In Fig. 2 is represented daily variation of turbulence coefficient at heights of 100, 500, and 1000 m. As can be seen from the figure, by 0900 hours turbulence extends to an insignificant height, with values of $K = 12-17 \text{ m}^2/\text{s}$ being attained at a height of 100 m. When at 100 m turbulence is developed, it starts to spread upwards. Although turbulence at 100 m starts to develop considerably earlier (around 0600), it attains maximum at all heights simultaneously from 1200 to 1300 hours.

The value of the turbulence coefficient during this time is $40 \text{ m}^2/\text{s}$ at 100 m , gradually dropping with height to $31 \text{ m}^2/\text{s}$ at 1000 m . Then the intensity of turbulence decreases and at 1433 K changes from $24 \text{ m}^2/\text{s}$ at $H = 100 \text{ m}$ to $10 \text{ m}^2/\text{s}$ at $H = 1000 \text{ m}$.

For more detailed consideration of this effect in Fig. 3 it is shown how the value of K changes with time at various heights in daylight hours.

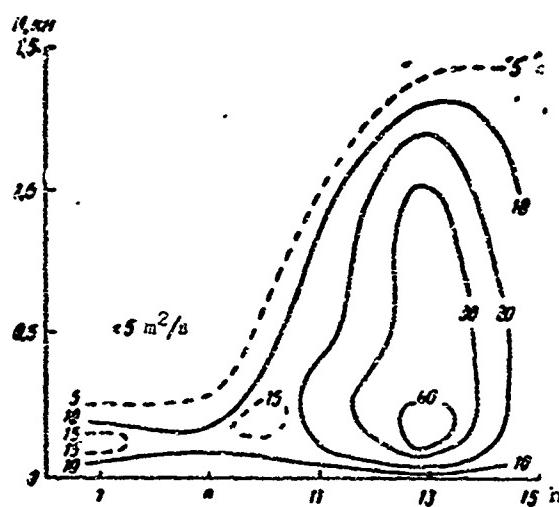


Fig. 3. Time cross section of the turbulence coefficient for 29 Sep 1946.

Unfortunately, there are no night observations. Apparently, at night, when in the lower part of the troposphere temperature stratification becomes stabler, the intensity of turbulence should decrease sharply.

4. Flights of 25 October 1946

Flights passed through the rear part of a cyclone, where secondary fronts, displaced to the south were observed. In the Moscow region were observed stratus clouds, weak snowfall at times, and northwest wind.

On this day four flights were made: at 0740, 0925, 1125, and 1335 hours. Platforms were made at lower and upper cloud boundaries.

At 0740 at heights of 100 and 170 m (at the lower cloud boundary) the aerologist noted moderate bumping of the Po-2 aircraft. The turbulence coefficient was $12 \text{ m}^2/\text{s}$ at 100 m and $15 \text{ m}^2/\text{s}$ at 250 m (Fig. 1d). At remaining heights flight was calm.

By 1128 turbulence had strengthened. The turbulence coefficient was considerable from the lower flight platform to the height of the lower St boundary, which was at 450 m. At 100 m K was $16 \text{ m}^2/\text{s}$, while at 300 m the value decreased to $11 \text{ m}^2/\text{s}$, and then increased at the lower St boundary to $18 \text{ m}^2/\text{s}$. The vertical temperature gradient in this layer varied from $0.33-0.78/100 \text{ m}$. The aerologist also noted moderate bumping at three lower levels. Strengthening of intensity of turbulence at the lower cloud boundary agrees well with data from work [4].

At 1335 overloads of the aircraft were observed only in the form of bursts, and it was not possible to calculate the value of the turbulence coefficient.

As a result of the foregoing the following preliminary conclusions can be made.

The turbulence coefficient under anticyclonic conditions has well-defined vertical profile. On it most frequently are noticeable one or two maxima, the values of which are determined by conditions of thermal stratification of the atmosphere.

Values of the turbulence coefficient have daily variation. Maximum of K is attained in the daytime hours at all heights simultaneously, but the beginning of diurnal variations is most clearly expressed for the height of 100 m.

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Submitted
15 August 1966

THEORY OF SPATIAL-TEMPORAL CORRELATION
OF VELOCITIES IN AN ISOTROPIC
TURBULENT FLOW

V. I. Smirnov and B. Sh. Shapiro

Conditions of homogeneity and isotropy, and also the hypothesis on quasi-normality of velocity distribution are used for derivation of an equation for the Eulerian two-point space-time correlation function in nonstationary turbulent flow, generalizing equation the Chandrasekhar for stationary turbulence. Solution of the derived equation for the last stage of degeneration of turbulence agrees with results obtained by other authors. A method of solving the Chandrasekhar equation by expanding the solution in power series is developed. Coefficients of the series are determined uniquely if space and time correlation functions are given. With rapid decrease of derivatives of the correlation function with distance, it is sufficient to assign the space correlation function. The time microscale of correlation is determined by the mean square of velocity, kinematic viscosity, and the space correlation function. There is a class of universal solutions, i.e., solutions not depending on viscosity and intensity of turbulence. Certain solutions not depending on the time interval are investigated.

1. Introduction

Turbulent flow might be simultaneously uniform, isotropic, and stationary (in a statistical sense) only in the presence within it of corresponding sources of energy, replenishing the decrease of energy of turbulence due to its dissipation. The theory of

turbulence is most simply built around the three above-mentioned conditions, and, bypassing the question of sources of energy, around introduction of certain more or less formal assumptions.

In 1955-1956 Chandrasekhar offered the deductive theory of turbulence [9, 10], based on postulates of homogeneity, isotropy, and stationarity, and also on the well-known hypothesis of quasi-normality (Millionshchikov hypothesis), associating correlation tensors of second and fourth ranks. On the basis of the Navier-Stokes equation for incompressible liquid Chandrasekhar derived a nonlinear partial differential equation, containing only one unknown scalar function, depending on space and time intervals. In view of the assumed difficulty of solving this equation in the general case, its author limited himself to finding a solution for the case when viscosity does not play a role and can be considered equal to zero, and arrived at the conclusion that theory does not contradict the statements of the well-known Kolmogorov theory on the inertial interval.

In developing the Chandrasekhar method, S. Panchev derived a dynamic equation for the space-time correlation function and an equation for other correlation functions [11].

Later A. I. Ivanovskiy and I. P. Mazin showed [2] that with application of the above-mentioned assumptions one more equation, different from the Chandrasekhar equation can be obtained. The form of the equation derived by Ivanov and Mazin was definitized in the work of V. I. Smirnov [5].

A check of the derivation of the Chandrasekhar equation, expounded in work [9], shows that in computations error in sign is allowed, after correction of which instead of the Chandrasekhar equation a pair of new equations is obtained. However, by modifying the derivation one can also obtain the Chandrasekhar equation. It is shown below that rejection of the condition of stationarity makes it possible to prove that this pair of new equations has no meaning.

In this work the Chandrasekhar equation is generalized for the case of nonstationary turbulence, solution of this equation for the last stage of degeneration of turbulence is shown, and certain solutions of the Chandrasekhar equation (for the case of stationary turbulence) are obtained.

2. Equation Describing Nonstationary Turbulence

Rejecting the condition of stationarity, adopted by Chandrasekhar [9], we will preserve conditions of homogeneity, isotropy, and incompressibility of liquid. Derivation of the equation will be a generalization of the Chandrasekhar derivation [9]. Let us introduce the following correlation tensors:

$$\begin{aligned}
 Q_{ij} &= \overline{u_i(\vec{r}', t') u_j(\vec{r}'', t'')} = \overline{\dot{u}_i \dot{u}_j}, \\
 T_{ij,k} &= \overline{u_i(\vec{r}', t') u_j(\vec{r}', t'') u_k(\vec{r}'', t'')} = \overline{\dot{u}_i \dot{u}_j \dot{u}_k}, \\
 P_{ij} &= \overline{u_i(\vec{r}', t') u_j(\vec{r}', t'') u''_k(\vec{r}'', t'')} = \overline{\dot{u}_i \dot{u}_j u''_k}, \\
 \bullet &= p/\rho, \\
 Q_{ij,k} &= \overline{u_i(\vec{r}', t') u_j(\vec{r}', t'') u_k(\vec{r}'', t'')} = \overline{\dot{u}_i \dot{u}_j \dot{u}_k \bullet}. \quad (1)
 \end{aligned}$$

Here \vec{r}' and \vec{r}'' - radius vectors of two points in space; t' and t'' - two moments of time; u_i - component of velocity vector along the i -th axis of the Cartesian coordinate system, in which $\bar{u}_i = 0$; p - pressure; ρ - density. Averaging is done by sets.

Because of isotropy and homogeneity, all correlation tensors depend only on $\vec{r} - \vec{r}' - \vec{r}''$, and also on t' and t'' (on $t = |t'' - t'|$ in the case of stationarity). In particular Q_{ij} and $T_{ij,k}$ have the following form [9]:

$$\begin{aligned}
 Q_{ij} &= \frac{Q'}{r} \delta_{ij} - (rQ' + 2Q)\delta_{ij}, \\
 T_{ij,k} &= \frac{2}{r} T \epsilon_{ijk} \epsilon_{ik} - (rT' + 3T)(\delta_{jk} + \delta_{kj}) + 2T \delta_{jk} \delta_{ij}, \quad (2)
 \end{aligned}$$

where $r = |\vec{r} - \vec{r}'|$; Q and T - determining scalars of tensors Q_{ij} and $T_{ij,k}$.

depending on r , t' and t'' ; δ_{ij} — Kronecker delta; primes near Q and T signify differentiation with respect to r .

The basic statistical hypothesis will be formulated as

$$Q_{ij,N} = Q_{ii}Q_{jj} + Q_{ii}Q_{jk} + Q_{ij}(0, 0, t', t')Q_{kj}(0, 0, t'', t''), \quad (3)$$

which is a further generalization of the hypothesis adopted by Chandrasekhar in work [9] and generalizing the Millionshchikov hypothesis.

Let us consider the Navier-Stokes equation for the first point (\vec{r}, t')

$$\frac{\partial u'_i}{\partial r'} + \frac{\partial}{\partial x_k} u'_i u'_k = - \frac{\partial u'}{\partial x_i} + \nabla^2 u'_j. \quad (4)$$

We multiply (4) by u'_j and average for the set

$$\frac{\partial Q_{ij}}{\partial r'} - \frac{\partial}{\partial x_k} T_{ikj} = \nabla^2 Q_{ij}. \quad (5)$$

Here by ∇^2 is understood $\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}$. Turning in (5) to determining scalars, we obtain the first equation for scalars

$$\left(\frac{\partial}{\partial r'} - \nabla D_s \right) Q = \left(r \frac{\partial}{\partial r} + 5 \right) T. \quad (6)$$

where D_n — Laplacian in n -dimensional space in case of spherical symmetry, equal to

$$D_n = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r}. \quad (7)$$

In order to find the second equation for scalars, we will start from the Navier-Stokes equation for the second point (\vec{r}', t'') :

$$\frac{\partial u''_i}{\partial r''} + \frac{\partial}{\partial x'_k} u''_i u''_k = - \frac{\partial u''}{\partial x'_i} + \nabla^2 u''_j. \quad (8)$$

We multiply (8) by $u'_i u'_k$ and average. We then find

$$\frac{\partial}{\partial r^j} T_{ik,j} + X_{ik,j} = r^2 T_{ik,j}, \quad (9)$$

where we designate

$$X_{ik,j} = \frac{\partial}{\partial t_i} Q_{ik,j} + \frac{\partial P_{ik}}{\partial j}. \quad (10)$$

Turning in (9) to determining scalars, we obtain the second equation for scalars

$$\left(\frac{\partial}{\partial r^i} - r D_i \right) T = - X, \quad (11)$$

where X — determining scalar of tensor $X_{ik,j}$.

As Chandrasekhar showed [9], from the hypothesis of form (3) and condition of isotropy it follows that

$$\frac{\partial}{\partial r} \left(5 + r \frac{\partial}{\partial r} \right) X = - 2Q \frac{\partial}{\partial r} D_i Q. \quad (12)$$

Generalization (3) in the case of nonstationarity does not hinder the obtaining of equation (12), where the form of the last term of the right side of (3), not depending on coordinates, does not play a role, inasmuch as in derivation of (12) we encounter only derivative $Q_{ik,j}$ with respect to t , see (10).

It is easy to check for the existence of identity

$$\left(5 + r \frac{\partial}{\partial r} \right) D_i = D_i \left(5 + r \frac{\partial}{\partial r} \right). \quad (13)$$

Applying operator $\left(\frac{\partial}{\partial r} - r D_i \right)$ to (6) and considering (13), we find

$$\left(\frac{\partial}{\partial r} - v D_s \right) \left(\frac{\partial}{\partial t'} - v D_t \right) Q = - \left(r \frac{\partial}{\partial r} + 5 \right) X, \quad (14)$$

whence with the help of (12) we finally obtain

$$\frac{\partial}{\partial r} \left(\frac{\partial}{\partial r'} - v D_s \right) \left(\frac{\partial}{\partial t'} - v D_t \right) Q = 2Q \frac{\partial}{\partial r} D_s Q. \quad (15)$$

We must emphasize that in this equation it is impossible to consider $t' = t''$, which follows from the derivation, see transition from (4) to (5) and from (8) to (9). Owing to this it is impossible to make the transition from (15) to the equation for $Q(r, t, t')$, derived by Pancher from the Karman-Howard equation with the help of the quasi-normality hypothesis [11].

In the particular case of stationary turbulence scalars depend only on r and $t = |t'' - t'|$, so that $\frac{\partial}{\partial r'} = -\frac{\partial}{\partial r}$. Let us assume for definitiveness that $t'' > t'$; then $\frac{\partial}{\partial r'} = \frac{\partial}{\partial t}$ and (15) becomes the Chandrasekhar equation for $Q(r, t)$:

$$\frac{\partial}{\partial r} \left(\frac{\partial^2}{\partial r^2} - v^2 D_s^2 \right) Q = -2Q \frac{\partial}{\partial r} D_s Q. \quad (16)$$

Thus for Q we obtain a unique equation (not counting the equation obtained by the other method, see below), when under the assumption of stationarity there can be obtained, besides equation (16), an additional pair of equations

$$\frac{\partial}{\partial r} \left(\frac{\partial}{\partial t} \pm v D_t \right)^2 Q = 2Q \frac{\partial}{\partial r} D_s Q. \quad (17)$$

if during derivation of the second equation for scalars we start with the Navier-Stokes equation for the first point (r', t'). Incidentally, it is exactly in this way that Chandrasekhar proceeds, but he allows error in sign (equality (23) in work [9]) and therefore arrives at equation (16). With nonstationarity it is impossible to start both times from the first point, since we then obtain, in

particular, different correlation tensors $\overline{u_i' u_k' u_j'}$ and $\overline{u_i' u_k' u_j'}$ with different determining scalars. Thus it is impossible here to obtain a nonstationary equation which would become (17) in the case of stationarity. Consequently, the pair of equations (17) is excluded. If, in contrast to (4), we start from the second point, we again obtain equation (15) for the nonstationary case and equation (16) for the stationary.

As was shown by Ivanovskiy and Mazin [2], and also by Smirnov [5], assumptions of homogeneity, isotropy, and stationarity permit obtaining one more equation, different from equation (16). Again rejecting the condition of stationarity, we leave on the left in equations (4) and (8) only the partial derivative with respect to time, multiply left and right sides and average:

$$\frac{\partial Q}{\partial t} = \Phi. \quad (18)$$

where $t' \neq t''$, as in (15), and Φ - is a function whose expression through Q has not yet been found. However, in the particular case of strongly degenerated turbulence, when in equations (4) and (5) it is possible to reject nonlinear members, the form of Φ is easy to find

$$\frac{\partial Q}{\partial t} = v^2 D_5^2 Q. \quad (19)$$

Ivanovskiy and Mazin consider [2] that nonlinear members can be rejected and for stationary turbulence also, if we limit ourselves to the viscous subdomain, where $Re \ll 1$. We then obtain [5]

$$\left(\frac{\partial^2}{\partial t^2} + v^2 D_5^2 \right) Q = 0, \quad (20)$$

which differs from (16) if we reject the nonlinear right part. Apparently, requirements of homogeneity and isotropy, together with the hypothesis of quasi-normality, are not so strong as to make it possible to derive a unique equation.

It is possible to doubt the legality of rejection of nonlinear members during derivation of equation (20), inasmuch as stationary, undamped turbulence exists as a result of nonlinear energy transfer from bigger vortexes to smaller. Therefore it would be desirable to obtain full nonlinear equation (18) and to study its solutions. Until this is done there is no possibility of selecting some one of the two equations (15) and (18) and stating that it, indeed, describes the structure of turbulent flow.

It is easily confirmed that equations (15) and (19) correctly describe the final period of degeneration of turbulence. If it is considered that at initial moment of time

$$Q(r, 0, 0) = \text{const} \exp\left(-\frac{r^2}{r_0^2}\right). \quad (21)$$

Then function

$$Q(r, t', t'') = \text{const} \left[1 + \frac{4\nu(t' + t'')}{r_0^2} \right]^{-\nu/2} \exp\left[-\frac{r^2}{r_0^2 + 4\nu(t' + t'')} \right] \quad (22)$$

satisfies, which is checked directly, both equation (15) with zero in the right part and equation (19). Solution of equation (22) for $t = t' = t''$ becomes solution $Q(r, t, t)$, found by L. G. Loytsyanskiy [3] for $t \rightarrow \infty$, when the form of initial condition (21) does not play a role

$$Q(r, t, t) = \text{const} t^{-\nu/2} \exp\left(-\frac{r^2}{8\nu t}\right). \quad (23)$$

and also solution of the Kármán-Howard with initial condition (21), found by Batchelor and Townsend [8].

Below will be examined certain solutions of the Chandrasekhar (15).

3. Solution of the Chandrasekhar Equation Depending on Time Interval

Let us examine equation (16). It is convenient to introduce dimensionless longitudinal correlation function $f(r, t)$. By definition

$$f = -\frac{2Q}{\sigma^2}. \quad (24)$$

Substituting (24) in (16), we obtain an equation for f

$$\frac{\partial}{\partial r} \left(\frac{\partial^2}{\partial t^2} - v^2 D_s^2 \right) f = \bar{u}^2 f \frac{\partial}{\partial r} D_s f. \quad (25)$$

Function $f(r, t)$ should be an even function of r and t owing to conditions of isotropy and stationarity [7]. Let us assume that $f(r, t)$ can be represented in the form of a power series

$$f(r, t) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} a_{lk} r^{2l} t^{2k}. \quad (26)$$

Putting this series in equation (16), we find

$$\begin{aligned} & \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} ik(2k-1)a_{ik} r^{2i-1} t^{2k-2} - 2v^2 \sum_{i=2}^{\infty} \sum_{k=0}^{\infty} i(i-1)(i-2)(4i^2+8i+3)a_{ik} \times \\ & \times r_{ik}^{i-3} t^{2k-2} - \bar{u}^2 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} l(l-1)(2l-3)a_{ik} a_{lm} r^{2i+2l-3} t^{2k+2m}. \end{aligned} \quad (27)$$

We divide this equation by r , make linear transformation of indices of summation in such a way that summation starts each time from the zero index, and equate coefficients of members with identical powers of r and t . Then we obtain.

$$\begin{aligned} a_{i+1, k+1} = & \frac{1}{(i+1)(k+1)(2k+1)} \left[2v^2(i+3)(i+2) \times \right. \\ & \times (i+1)(4i^2+32i+63)a_{i+2, k} + \bar{u}^2 \sum_{p=0}^i \sum_{q=0}^k (p+2) \times \\ & \left. \times (p+1)(2p+7)a_{p+2, q} a_{i-p, k-q} \right]. \end{aligned} \quad (28)$$

where $i, k > 0$.

Let us write out the matrix of coefficients a_{ik} :

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0k} & \dots \\ a_{10} & a_{11} & a_{12} & \dots & a_{1k} & \dots \\ a_{20} & a_{21} & a_{22} & \dots & a_{2k} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nk} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{vmatrix}. \quad (29)$$

According to (28), coefficients with value of second (temporal) index > 1 depend on a finite number of coefficients with lower value of second index, i.e., each column of matrix (29), in addition to elements of the first line, is a function of elements of columns to the left of the examined column, and also of elements of the first line. Consequently, if we assign all elements of form a_{ni}, a_{ki} , ($i, k > 0$), then by formula (28) all coefficients a_{ik} are uniquely determined. In other words, the Chandrasekhar equation (25) uniquely determines the space-time correlation function $f(r, t)$, if space $f(r, 0)$ and time $f(0, t)$ correlation function are given.

If functions $f(r, 0)$, $f(0, t)$ are assigned graphically, they can be approximated by power series, and we can thereby determine some quantity of first coefficients a_{ni}, a_{ki} . If K 's of first coefficients a_{ki} are given, $2K$ coefficients a_{ni} are necessary in order to determine all a_{ik} to the left of straight line in (29).

In the particular case of $k = 0$ (second column of (29)) we find from (28):

$$a_{i+1,0} = \frac{1}{(i+1)} \left[2r^2(i+3)(i+2)(i+1)(4i^2+32i+63) \times \right. \\ \left. \times a_{i+2,0} + \bar{u}^2 \sum_{p=0}^i (p+2)(p+1)(2p+7) a_{p+2,0} a_{i-p,0} \right]. \quad (30)$$

i.e., all elements of form a_{ni} , with the exception of a_{0i} , are determined only by elements of form a_{ni} . Where $i = 0$ we have the simple formula

$$a_{01} = 14(\bar{u}^2 a_{20} + 54 \bar{v}^2 a_{30}), \quad (31)$$

experimental check of which would be interesting.

With sufficiently rapid decrease of derivatives $f(r, t)$ with distance for unique determination of $f(r, t)$ one longitudinal space correlation function $f(r, 0)$ is sufficient.

If the following condition is met

$$\left(\frac{\partial}{\partial r} - r^2 D_s^2 \right) f(r, t) \Big|_{r=0} = 0, \quad (32)$$

which seems natural, one can determine all coefficients of expansion of $f(0, t)$, a_{0k} , i.e., the function $f(0, t)$. Integrating (25) with respect to r , we find

$$\left(\frac{\partial}{\partial r} - r^2 D_s^2 \right) f = -\bar{u}^2 \left[C(t) - \int f(r', t) \frac{\partial}{\partial r'} D_s f(r', t) dr' \right], \quad (33)$$

where

$$C(t) = \int f(r', t) \frac{\partial}{\partial r'} D_s f(r', t) dr' = \sum C_k r^k. \quad (34)$$

If we substitute in (33) $f(r, t)$ in form (26), then, besides (28), we obtain

$$a_{0,k+1} = \frac{200 a_{0k} - \bar{v}^2 C_k}{2(k+1)(2k+1)}, \quad k > 0. \quad (35)$$

These formulas together with (28) make it possible to successively determine all a_{0k} , $k > 0$. Actually a_{0k} equals

$$a_{0k} = \frac{200 a_{00} - \bar{v}^2 C_k}{2}, \quad (36)$$

$$C_0 = \int f(r, 0) \frac{\partial}{\partial r} D_0 f(r, 0) dr. \quad (37)$$

For C_k it is easy to obtain the formula

$$C_k = 4 \sum_{p=0}^k \int \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (i+1)(i+2)(2i+7) a_{i+2,p} a_{i,k-p} r^{i+2+p} dr. \quad (38)$$

As was shown above, from the first column of the matrix of coefficients (29) we can determine the second. Further, with the help of (28) from elements of the second column we determine all elements of the third but a_{30} . The a_{30} elements we find from (35), inasmuch as element a_{30} has been found, while C_1 can be found from (38), since in the right part of (38) are only elements of form a_{ip} , $p = 0, 1$, which have already been determined. Analogously we find elements of the fourth column from elements of the third, etc.

Let us consider formula (36). If $f(r, 0)$ and \bar{u}^2 , (36) determines the Eulerian temporal microscale of turbulence τ_0 , equal to

$$\tau_0 = \sqrt{-\frac{1}{a_{30}}}. \quad (39)$$

Since as, obviously, we must have $a_{30} < 0$, from (36) we obtain the lower limit of energy of stationary turbulent flow;

$$\bar{u}^2 > \frac{f_{200}^2 a_{30}}{C_0}. \quad (40)$$

Experimental check of formulas (36) and (40) is, from our point of view, of great interest.

Let us consider the example when $f(r, 0)$ has the form

$$f(r, 0) = e^{-\frac{r^2}{L^2}}. \quad (41)$$

We easily find that

$$a_{10} = -\frac{1}{r_0^2}, \quad a_{20} = \frac{a_{10}^2}{2}, \quad C_0 = -6a_{10}.$$

Then from (36) and (39) we obtain

$$\tau_0^2 = \frac{1}{-s_{10}(3r_0^2 + 70^2 a_{10})}. \quad (42)$$

As appraisals founded on experimental data for turbulence with large values of Re show, usually [6]

$$\frac{70^2 |a_{10}|}{3r_0^2} = \frac{7}{9} \frac{\epsilon}{(\bar{u}^2)} < \frac{7}{9 \cdot 600} \ll 1. \quad (43)$$

where ϵ is rate of dissipation of energy of flow. Here we use the well-known expression for τ_0^2 [7]

$$\tau_0^2 = \frac{30\bar{u}^2}{\epsilon}. \quad (44)$$

Then from (42) we obtain

$$\tau_0^2 \approx \frac{r_0^2}{3r_0^2} = \frac{10r_0}{\epsilon}. \quad (45)$$

Let us note that in examined example formula (40) is satisfied according to (43).

It is known that correlation function $f(r, 0)$ is poorly described by a formula of form (41), so that relationships (42) and (45) have to be very approximate. On the other hand, from the expression for C_0 (37) it follows that C_0 is determined basically by small distances, where $f(r, 0)$ and its derivatives are not very small. Therefore, there is reason to think that (42) and (45) correctly give the order of magnitude of τ_0 . In the theory of turbulent coagulation of aerosols is obtained the formula [6]

$$\tau_0^2 \approx 35 \frac{r_0^2}{\epsilon}. \quad (46)$$

Possibly, the use for $f(r, 0)$ of a more exact expression than (41) would allow us to achieve better agreement of both formulas, or even make it possible to definitize (46).

When $\varepsilon = 10 \text{ cm}^2/\text{s}^3$ and $\nu = 0.14 \text{ cm}^2/\text{s}$, we find from (45) $\tau_0 = 0.37 \text{ s}$, which is realistic.

We showed that during fulfillment of condition (32) the Chandrasekhar equation (25) for space-time correlation function $f(r, t)$ has unique solution in the class of functions expandable in power series in even degrees of r and t , completely determined by assignment of space correlation function $f(r, 0)$. Solution of $f(r, t)$ can be found by the formulas derived above. The form of $f(r, 0)$ should be determined independently either from theory or from experiment. Furthermore, the mean square of the velocity component $\bar{u^2}$ and kinematic viscosity ν have to be known.

Thus the Chandrasekhar equation is compatible with the well-known Kolmogorov theory, or in general with any theory giving $f(r, 0)$, if, of course, it does not contain propositions contradicting conditions of homogeneity, isotropy, and stationarity, or the hypothesis on quasi-normality of velocity distribution.

Certain theoretical results show that the hypothesis of quasi-normality cannot be fulfilled exactly (see, for example, work [4]). However, it would hardly be correct to interpret such results as proof of the total unfoundedness of any derivation obtained through the use of this hypothesis. It is known that the system of equations for correlation functions is infinite, so that in any conceivable theory of turbulence it will be necessary to adopt some or other hypotheses in order to be limited to a finite number of equations, and it is fully possible that any of these hypotheses will lead to more or less considerable contradictions and noncorrespondences.

4. Universal Solutions of the Chandrasekhar Equation

Let us examine a special family of solutions of the Chandrasekhar equation for which we will turn in equation (25) to dimensionless variables

$$r = \sqrt{\frac{2t}{\bar{u}^2}}, \quad \tau = \frac{t}{\bar{u}^2}. \quad (47)$$

where the coefficient 2 is introduced for considerations of convenience, mentioned below, see formulas (102) and (103). Then we have, assuming that f depends on ρ and τ ,

$$\frac{\partial}{\partial \rho} \left(\frac{\partial}{\partial \rho} - D_0^2 \right) f = 2f \frac{\partial}{\partial \rho} D_0 f. \quad (48)$$

This equation does not include quantities ν and \bar{w}^2 , characterizing liquid and turbulence. Therefore under corresponding boundary conditions it gives universal solutions. Instead of (28) we obtain

$$c_{i+1, k+1} = \frac{2}{(i+1)(k+1)(2k+1)} [(i+3)(i+2)(i+1) \times \\ \times (4i^2 + 32i + 63) c_{i, k, i, k} + \sum_{p=0}^i \sum_{q=0}^k (p+2)(p+1)(2p+7) c_{p+2, q} c_{i-p, k-q}], \quad (49)$$

where c_n — coefficients in expression

$$f(\rho, \tau) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} c_{ik} \rho^i \tau^k. \quad (50)$$

If a condition analogous to (32) is met, relationships of type (35) and (36) apply, where the latter has the form

$$c_{ik} = 340 c_{i0} - C_i. \quad (51)$$

However, universal solutions contradict the Kolmogorov theory actually from comparison of (50) and (44) we easily find

$$\bar{w}^2 = \sqrt{\frac{(-z_{10})}{15}} c_{10}, \quad (52')$$

where z_{10} is a dimensionless quantity, while from the Kolmogorov theory it follows that

$$\bar{w}^2 \sim (eL)^{5/3}. \quad (53)$$

where L — magnitude of large-scale vortexes. Apparently, universal solutions are of little interest from a physical standpoint.

5. Solutions not Depending on Time Interval

Equation (25) also has solution not depending on time interval, the study of which is of certain, basically metrical, interest. Let us record stationary equation (25) in the form

$$-\nu^2 \frac{\partial}{\partial r} D_0^2 f = \bar{u}^2 f \frac{\partial}{\partial r} D_3 f. \quad (54)$$

Representing $f(r)$ in the form of a series

$$f(r) = \sum_{i=0}^{\infty} a_i r^i, \quad a_0 = 1, \quad (55)$$

we find

$$a_{i+3} = -\frac{\bar{u}^2}{2\nu^2} \frac{\sum_{p=0}^i (p+2)(p+1)(2p+7) c_{p+2} a_{i-p}}{(i+3)(i+2)(i+1)(4i^2 + 32i + 63)}, \quad (56)$$

where $i \geq 0$. If a_1 and a_2 are assigned, formula (56) permits finding all other a_k , $k > 3$.

If conditions (32) is met, with the help of formula (35), where we consider $a_{k,k+1} = 0$, we can determine a_2 ,

$$a_2 = \frac{\bar{u}^2 C_0}{280\nu^2}, \quad (57)$$

so that only a_1 is arbitrary.

Equation (44) also has a particular solution of special kind. In order to find it we will start with equation (25), containing time.

Let us integrate both parts of equation (25) from r to ∞ and assume that (32) is satisfied. We then obtain:

$$\left(\frac{\partial^2}{\partial t^2} - \nu^2 D_0^2 \right) f = -\bar{u}^2 \int_r^\infty f(r', t) \frac{\partial}{\partial r'} D_3 f(r', t) dr'. \quad (58)$$

We will represent solution of equation (58) in the form

$$f(r, t) = f_0(r, t) + \bar{u}^2 g(r, t), \quad (59)$$

where f_0 satisfies equation

$$\frac{\partial f_0}{\partial r} - v^2 D_s^2 f_0 = 0, \quad (60)$$

with boundary condition

$$f_0(r, 0) = f(r, 0), \quad (61)$$

and g satisfies equation

$$\frac{\partial g}{\partial r} - v^2 D_s^2 g = - \int_r^\infty f(r', t) \frac{\partial}{\partial r'} D_s f(r', t) dr', \quad (62)$$

with boundary condition

$$g(r, 0) = 0. \quad (63)$$

Let us find the solution for (60), using apparatus of the δ -function theory, described in work [1], similarly to the way in which this is done in work [2]. Let us represent f_0 in the form

$$f_0(r, t) = f(r, 0) + \varphi(r, t), \quad (64)$$

where φ satisfies equation

$$\frac{\partial \varphi}{\partial r} - v^2 D_s^2 \varphi = v^2 D_s^2 f(r, 0) \quad (65)$$

with boundary condition

$$\varphi(r, 0) = 0. \quad (66)$$

The solution of (65) is presented in the form

$$\varphi(r, t) = - \iint v^2 D_s^2 f(r') \delta(|\vec{r}' - \vec{r}|, |t' - t|) dv' dt'. \quad (67)$$

Here G — Green's function, $d\sigma'$ — element of volume in five-dimensional space

$$d\sigma' = r^4 \sin^2 \chi \sin^2 \Psi \sin \theta dr' d\lambda d\Psi d\phi, \quad (68)$$

where r' — "distance to origin of coordinates," and χ, Ψ, θ and ϕ — "angles" formed by radius vector with coordinate axes [1]. Integration is carried out over the entire infinite "space."

Green's function satisfies equation [1].

$$\frac{\partial G}{\partial t} - \nabla^2 G = -\delta(|\vec{r}' - \vec{r}|, |t' - t|), \quad (69)$$

using which, we easily obtain from (67)

$$f(r, t) = -\frac{\partial}{\partial t} \iiint f(r', 0) \Omega(|\vec{r}' - \vec{r}|, |t' - t|) d\sigma' dt' - f(r, 0). \quad (70)$$

then from (64) we have

$$\begin{aligned} f_0(r, t) &= -\frac{\partial}{\partial t} \iiint f(r', 0) \Omega(|\vec{r}' - \vec{r}|, |t' - t|) d\sigma' dt' = \\ &= -\frac{\partial}{\partial t} \iint f(r', 0) \Omega(|\vec{r}' - \vec{r}|, t') d\sigma'. \end{aligned} \quad (71)$$

The singular part of Green's function is equal to

$$\begin{aligned} G_1(r, t) &= -\frac{1}{(2\pi)^5} \int dk \int dX \int d\sigma' \int d\Theta \int d\phi \int d\lambda \\ &\times \left[\frac{e^{i(kt + k\lambda r \cos^2 \chi + \sin^2 \chi \sin^2 \Psi \sin \theta)}}{-\omega^2 - k^2} \right]. \end{aligned} \quad (72)$$

Let us assume that Green's function G is different from 0 only when $t > 0$. Therefore we will assume

$$G(r, t) = G_1(r, t) + \frac{i}{|t|} G_1(r, t) \begin{cases} 2G_1(r, t); & t > 0 \\ 0; & t < 0. \end{cases} \quad (73)$$

As a result we find for $t > 0$

$$G(r, t) = -\frac{\frac{1}{2}}{(2\pi)^{\frac{3}{2}} t^{\frac{1}{2}}} \left[\frac{1}{r^2} \int_0^t e^{-\frac{r^2}{4t'}} dt' - \frac{1}{r} e^{-\frac{r^2}{4t}} \right]. \quad (74)$$

Condition (73) together with the condition of parity of f_0 and g with respect to t mean that the sought solution of $f(r, t)$ does not have to depend on t . Below we will confirm that this is indeed the case.

Let us represent $f(r, 0)$ in the form

$$f(r, 0) = 1 + a_{10}r^2 + a_{20}r^4 + \dots = \sum_{n=0}^{\infty} a_{n0}r^{2n}; a_{00} = 1. \quad (75)$$

The series contains only even degrees of r owing to isotropy [7] and is assumed uniformly convergent everywhere.

We rewrite (71) in the form

$$f_0(r, t) = -\frac{1}{(2\pi)^{\frac{3}{2}} t^{\frac{1}{2}}} \int_0^t f(r', 0) K(r', r) dr', \quad (76)$$

where K is found after integration with respect to angles

$$\begin{aligned} K(r', r) = & \frac{2}{3} \left[\frac{1}{r'} \left(\int_0^{r'} e^{-tr'} dt' - \int_0^r e^{-tr'} dt' \right) + \right. \\ & + \frac{1}{r'^2} \left(\int_0^{r'} e^{-tr'} dt' + \int_0^r e^{-tr'} dt' \right) + \frac{1}{3r'} \left[\frac{1}{r'} (e^{-tr'+r'} - e^{-tr-r'}) - \right. \\ & \left. \left. - \frac{r}{r'} (e^{-tr'+r'} + e^{-tr-r'}) + \frac{(r-\frac{1}{2})}{r'^2} (e^{-tr'+r'} - e^{-tr-r'}) \right] \right]. \end{aligned} \quad (77)$$

where

$$\theta = \frac{1}{4\pi}. \quad (78)$$

Integrating in (76), we find

$$f_0(r, t) = \frac{2\pi}{3} \frac{\partial^{\frac{1}{2}}}{\partial p^{\frac{1}{2}}} \left\{ 2E_0 + \sum_{n=0}^{\infty} a_{2n} \left[\left(\frac{1}{p} - \frac{1}{2pr^2} \right) U_{2n+1} - \left(\frac{2}{2n+3} + \frac{1}{p^2} \right) U_{2n+2} + \frac{U_{2n+3}}{p^2} - \frac{2U_{2n+5}}{r^2(2n+5)} \right] \right\}, \quad (79)$$

where E_0 — integration constant, proportional to $p^{-\frac{1}{2}}$, and $U_k(r, p)$ — functions, equal to

$$U_k = \int r^k [e^{-kr+p} + (-1)^k e^{-kr-p}] dr. \quad (80)$$

Integration gives

$$\begin{aligned} U_{2n} &= 2\pi^{\frac{1}{2}} p^{\frac{1}{2}} \sum_{l=0}^n C_{2n}^{2l} r^{2l} \frac{(2n-1-2l)!!}{2^{n+l-1} p^{n+l-1}} \\ U_{2n+1} &= -2\pi^{\frac{1}{2}} p^{\frac{1}{2}} \sum_{l=0}^n C_{2n+1}^{2l+1} r^{2l+1} \frac{(2n-1-2l)!!}{2^{n+l-1} p^{n+l-1}}. \end{aligned} \quad (81)$$

Substituting (81) in (79), we find an expression for f_0 in the form of a series, the structural rule for which can be seen if we write out several of the first members:

$$\begin{aligned} f_0(r, t) &= 1 + a_{10} \left(\frac{5}{5} + r^2 \right) + a_{20} \left(\frac{5 \cdot 7}{2 \cdot 2r^2} + 2 \frac{7r^2}{23} + r^4 \right) + \\ &\quad + a_{30} \left(\frac{5 \cdot 7 \cdot 9}{2 \cdot 2 \cdot 2r^2} + 3 \frac{7 \cdot 9r^2}{2 \cdot 2r^2} + 3 \frac{9r^4}{23} + r^6 \right) + \\ &\quad + a_{40} \left(\frac{5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 2 \cdot 2 \cdot 2r^2} + 4 \frac{7 \cdot 9 \cdot 11r^2}{2 \cdot 2 \cdot 2r^2} + 6 \frac{9 \cdot 11r^4}{2 \cdot 2r^2} + 4 \frac{11r^6}{23} + r^8 \right) + \dots \end{aligned} \quad (82)$$

The coefficient before the term with ordinal number i in parentheses, ahead of which stands coefficient a_m , is equal to the binomial coefficient C_i .

Let us return to expression (59) for solution of the Chendrasekhar equation (25). We will look for $g(r, t)$ from (62) by the method of successive approximations, substituting instead of $f(r', t)$ a series with unknown coefficients

$$f(r, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_{km} r^m t^k = \sum_{m=0}^{\infty} a_m r^m + \Phi(r, t), \quad (83)$$

where $\Phi(r, t)$ — is an even function of t and r . Parity of function $f(r, t)$ with respect to t follows from stationarity [7].

Let us examine a certain approximation, for which

$$f(r, t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{kl} r^{2k} t^l; \quad c_{kl} = a_{kl}. \quad (84)$$

If coefficients c_{kl} are known, then $g(r, t)$ can be recorded in the form

$$g(r, t) = \iiint_{r'} \left[\int f(r', t') \frac{\partial}{\partial r'} D_t f(r', t') dr' \right] \times \\ \times G(|\vec{r}' - \vec{r}|, |t' - t|) dv' dr', \quad (85)$$

where G — Green's function (74). Let us replace variables

$$g(r, t) = \iiint_{r'} \left[\int f(r', 0) \frac{\partial}{\partial r'} D_t f(r', 0) dr' \right] G(|\vec{r}' - \vec{r}|, \theta) dv' d\theta. \quad (86)$$

Assuming that series (84) can be differentiated and integrated term-by-term, we find:

$$\int_r^{\infty} f(r, 0) \frac{\partial}{\partial r} D_t f(r, 0) dr = C(\theta) - \int_0^r f(r, 0) \frac{\partial}{\partial r} D_t f(r, 0) dr = \\ = C(\theta) - \sum_{k=2}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{2k(2k+3)(2k-2)}{2k+2m-2} c_{mk} c_{nl} r^{2k+2m-2} \theta^{2n+2l}, \quad (87)$$

where

$$C(\theta) = \int_0^r f(r, 0) \frac{\partial}{\partial r} D_t f(r, 0) dr = \sum_{p=0}^{\infty} C_p \theta^{2p}, \quad (88)$$

where coefficients C_p are still unknown.

Let us find auxiliary function $W(r, \theta)$

$$W(r, \theta) = \int_0^r \left[\int_{r'}^{\infty} f(r', 0) \frac{\partial}{\partial r'} D_t f(r', 0) dr' \right] K(r', r; r') dr'. \quad (89)$$

Knowing $W(r, \theta)$, we can find $g(r, t)$ by the formula

$$g(r, t) = \frac{1}{4\pi^{\frac{1}{2}} r^{\frac{3}{2}}} \int_0^r \frac{1}{\theta^{\frac{1}{2}}} W(r, \theta) d\theta. \quad (90)$$

Substituting $f(r, t)$ here from (84), we find

$$\begin{aligned} W(r, \theta) = & \frac{1}{5} \left\{ (-2\sqrt{\pi}) \left[\left(6r^{\frac{3}{2}} \theta^{\frac{3}{2}} + \frac{3}{5} r^{\frac{1}{2}} \theta^{2\frac{1}{2}} \right) C(\theta) + \right. \right. \\ & \left. \left. + 2r^{\frac{1}{2}} \theta^{\frac{1}{2}} D(\theta) \right] + \sum_{k=2}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{klmn} \right\}. \end{aligned} \quad (91)$$

Here

$$\begin{aligned} P_{klmn} = & \frac{2k(2k+3)(2k-2)}{2k+2m-2} C_{kl} C_{mn} l^{2n+2l} \times \\ & \times \left[\frac{2U_{2k+2m+3}}{r^2(2k+2m+3)} + \left(\frac{2}{2k+2m} + \frac{1}{\beta r^2} \right) U_{2k+2} - \frac{1}{\beta r^2} U_{2k+2m+1} - \right. \\ & \left. - \frac{1}{\beta r^2} \left(r^2 - \frac{1}{2\beta} \right) U_{2k+2m-1} \right]. \end{aligned} \quad (92)$$

where function $U_k(r, \beta)$ is determined by (80), and

$$\begin{aligned} D(\theta) = & -\lim_{r \rightarrow \infty} \left[\frac{C(\theta) r^2}{2} - \sum_{k=2}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \times \right. \\ & \left. \times \frac{2k(2k+3)(2k-2)}{(2k+2m-2)(2k+2m)} C_{kl} C_{mn} r^{2k+2m} l^{2l+2n} \right]. \end{aligned} \quad (93)$$

It is easy to verify that the last expression equals

$$D(\theta) = - \int_0^{\infty} \left\{ \int_0^{\infty} f(r', \theta) \frac{\partial}{\partial r'} D_s f(r', \theta) dr' \right\} r' dr = \sum_{s=0}^{\infty} D_s \delta^{sp}. \quad (94)$$

It is possible to record P_{klmn} in the form

$$P_{klmn} = c_{kl} r^2 e^{2k+2l} C_{km}, \quad (95)$$

where

$$Q_{2m} = \frac{2k(2k+3)(2k-2)}{2k+2m-2} \left[\frac{2U_{2k+2m+3}}{r^2(2k+2m+3)} + \left(\frac{2}{2k+2m} + \frac{1}{pr^2} \right) U_{2k+2} - \right. \\ \left. - \frac{1}{pr^2} U_{2k+2m+1} - \frac{1}{pr^2} \left(r^2 - \frac{1}{2k} \right) U_{2k+2m-1} \right]. \quad (96)$$

We will first determine members of the sum from (87), enclosing members with the same highest degree of θ in parentheses

$$\sum_{k=2}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{klmn} = (P_{2000}) + (P_{2000} + P_{2010}) + \\ + (P_{4000} + P_{3010} + P_{2020} + P_{2100} + P_{2001}) + \\ + (P_{6000} + P_{5010} + P_{4020} + P_{3030} + P_{3100} + P_{2001}) + \dots = \\ = (a_{20}Q_{20}) + (a_{20}Q_{20} + a_{20}a_{10}Q_{21}) + \\ + (a_{40}Q_{40} + a_{30}a_{10}Q_{31} + a_{20}^2Q_{22} + a_{21}^2Q_{20} + a_{20}a_{01}^2Q_{30}) + \\ + (a_{60}Q_{60} + a_{50}a_{10}Q_{51} + a_{30}a_{20}Q_{32} + a_{30}a_{20}Q_{23} + a_{31}^2Q_{30} + a_{30}a_{01}^2Q_{30}) + \\ + \dots = [a_{20}Q_{20}] + \left[\left(\overline{a_{20} + \frac{2 \cdot 7 \cdot 1}{3 \cdot 9 \cdot 2} a_{10}a_{10}} \right) Q_{20} \right] + \\ + \left[\left(\overline{a_{40} + \frac{3 \cdot 9 \cdot 2}{4 \cdot 11 \cdot 3} a_{10}a_{20} + \frac{2 \cdot 7 \cdot 1}{4 \cdot 11 \cdot 3} a_{20}^2} \right) Q_{40} + (a_{21} + a_{20}a_{01}) \theta^2 Q_{20} \right] + \\ + \left[\left(\overline{a_{60} + \frac{4 \cdot 11 \cdot 3}{5 \cdot 13 \cdot 4} a_{10}a_{30} + \frac{3 \cdot 9 \cdot 2}{5 \cdot 13 \cdot 4} a_{20}a_{20} + \frac{2 \cdot 7 \cdot 1}{5 \cdot 13 \cdot 4} a_{30}a_{20}} \right) Q_{60} + \right. \\ \left. + (a_{21} + a_{30}a_{01}) \theta^2 Q_{30} \right] + \dots. \quad (97)$$

where

$$Q_{20} = \frac{6\sqrt{x}}{\frac{1}{p^2}} \left(\frac{5 \cdot 7}{2 \cdot 2 \cdot p^2} + 2 \frac{7r^2}{2p} + r^4 \right), \\ Q_{40} = \frac{6\sqrt{x}}{\frac{1}{p^2}} \left(\frac{5 \cdot 7 \cdot 9}{2 \cdot 2 \cdot 2 \cdot p^2} + 3 \frac{7 \cdot 9 r^2}{2 \cdot 2 \cdot p^2} + 3 \frac{9 r^4}{2p} + r^6 \right), \\ Q_{60} = \frac{6\sqrt{x}}{\frac{1}{p^2}} \left(\frac{5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 2 \cdot 2 \cdot 2 \cdot p^2} + 4 \frac{7 \cdot 9 \cdot 11 \cdot r^2}{2 \cdot 2 \cdot 2 \cdot p^2} + 6 \frac{9 \cdot 11 \cdot r^4}{2 \cdot 2 \cdot p^2} + 4 \frac{11 r^6}{2p} + r^8 \right). \quad (98)$$

.....

Let us note that expressions in parentheses in (98) and (82) accurately coincide, as it should be according to (79) and (96).

We substitute (97) in (91), then substitute the obtained expression for $W(r, \theta)$ in (90). After integration and substitution of $g(r, t)$ and $f_0(r, t)$ from (81) in (63)

$$\begin{aligned}
f(r, t) = & f(r, 0) + 2\eta(1 \cdot 5a_{10} + 2 \cdot 7a_{20}r^2 + 3 \cdot 9 \cdot a_{30}r^4 + \\
& + 4 \cdot 11a_{40}r^6 + 5 \cdot 13 \cdot a_{50}r^8 + \dots) + \\
& + 4r^2P(1 \cdot 5 \cdot 7 \cdot a_{10} + 3 \cdot 7 \cdot 9a_{20}r^2 + 6 \cdot 9 \cdot 11a_{30}r^4 + 10 \cdot 11 \cdot 13a_{40}r^6 + \dots) + \\
& + \bar{a}^2 \left[t \left(-\frac{D_1}{3r} - \frac{1}{10} \frac{C_0}{r^2} + \frac{a_{20}}{r^4} + \frac{b_2}{r^6} + \frac{b_4}{r^8} + \frac{b_6}{r^{10}} + \dots \right) + \right. \\
& \left. + R^2 \left(-\frac{C_0}{2} + 2 \cdot 7a_{20}r^2 + 3 \cdot 9b_2r^4 + 4 \cdot 11b_4r^6 + 5 \cdot 13 \cdot b_6r^8 + \dots \right) + O(R) \right]. \tag{99}
\end{aligned}$$

From the condition of stationarity of turbulence it follows that odd degrees of t have to drop from $f(r, t)$, see (83), where this requirement satisfies each of the approximations (see (84)). Equating coefficients of members containing t to zero, we find

It is easy to verify that the formulas of (100), starting from expression for a_{30} , are equivalent to (56), and that the formula for a_{20} coincides with (57).

If expressions for a_{k0} are substituted in (99), it will appear that all coefficients on members containing t^2 , turn into zero, and

thus, expansion of $f(r, t)$ in series in degrees of t has the form

$$f(r, t) = f(r, 0) + O(t^2). \quad (101)$$

Examining (100), one readily sees that all coefficients a_{k0} are proportional to

$$\frac{1}{\pi} = \frac{\pi^2}{2^3}. \quad (102)$$

Relationships of (100) constitute recursion formulas where the last coefficient is obtained as the sum preceding products, plotted in accordance with defined rule. It is easy to see that the factor $\frac{1}{\pi}$ enters in expression for a_{k0} in degree k . Consequently, series (75) can be rewritten in simpler form by introducing instead of r the dimensionless coordinate

$$p = \frac{r}{\xi}. \quad (103)$$

what corresponds to (36). Then instead of (65) we have

$$f(p, 0) = 1 + a_1 p^2 + a_2 p^4 + \dots = \sum_{k=0}^{\infty} a_k p^{2k}, \quad (104)$$

$$a_0 = 1.$$

where coefficients a_k are determined from recursion formulas:

$$\begin{aligned} a_1 &= -\frac{\zeta}{1 \cdot 3}, \quad \zeta = -\frac{5}{3}, \\ a_2 &= -\frac{\gamma}{2 \cdot 7}, \quad \gamma = -\frac{9}{10}, \\ a_3 &= -\frac{\beta_3}{3 \cdot 9}, \quad a_2 = \beta_2, \\ a_4 &= -\frac{\beta_4}{4 \cdot 11}, \quad \beta_3 = a_3 + \frac{2 \cdot 7 \cdot 1}{3 \cdot 9 \cdot 2} a_1 a_2. \end{aligned} \quad (105)$$

.....

$$a_n = -\frac{\beta_{n-1}}{\pi(2n+3)}, \quad \beta_n = \sum_{p=2}^n z_{np} a_p a_{n-p},$$

$$z_{np} = \frac{p(2p+3)(p-1)}{\pi(2n+3)(n-1)}.$$

Expressions for δ_0 and σ_0 can be found by switching from r to p in formulas (88) and (94):

$$\delta_0 = - \int_0^\infty \left\{ \int_0^\infty f(p') \frac{\partial}{\partial p'} D_0 f(p') dp' \right\} p dp. \quad (106)$$

$$\sigma_0 = - \int_0^\infty f(p) \left[\frac{\partial}{\partial p} D_0 f(p) \right] dp. \quad (107)$$

Introducing dimensionless time τ according to (47), we can record (58) in dimensionless variables:

$$\left(\frac{\partial}{\partial r^2} - D_0^2 \right) f = - 2 \int_0^\infty f(p', \tau) \frac{\partial}{\partial p'} D_0 f(p', \tau) dp'. \quad (108)$$

From (106), with the help of (108), we find a simpler expression than (106)

$$\delta_0 = - \frac{1}{2} \int_0^\infty p D_0^2 f(p) dp. \quad (109)$$

where the following relationship was used

$$\left. \frac{\partial^2 f(r, t)}{\partial r^2} \right|_{r=0} = \left. \frac{\partial^2 f(p, \tau)}{\partial p^2} \right|_{p=0} = 0. \quad (110)$$

following from (101).

With the help of (110) it is easy to show that $f(p, \tau)$, in general, does not depend on τ . Actually, by differentiating (108) twice with respect to τ , we find

$$\left. \frac{\partial^2 f}{\partial \tau^2} \right|_{\tau=0} = \left. \frac{\partial^2 f}{\partial r^2} \right|_{r=0} = \dots = 0, \quad (111)$$

i.e.,

$$f(p, \tau) = f(p, 0). \quad (112)$$

Relationships (105), (107), and (109) make it possible to find all coefficients a_k , inasmuch as ζ and η , completely determining a_k ,

are themselves determined by all α_k . The value of α_k should decrease rapidly with increase of k , so that it is sufficient to know a certain quantity of first α_k for a given degree of accuracy in determination of α_k and $f(\rho, 0)$.

Coefficients α_k were calculated on the "Minsk-1" computer. In integrals (106) and (109) instead of ∞ a certain ρ_m was selected which then was increased gradually.

We selected $\zeta^{(0)}$ and $\eta^{(0)}$ arbitrarily and then in first approximation obtained $\zeta^{(1)}(\rho_m)$, $\eta^{(1)}(\rho_m)$, etc., until in a certain n -th approximation changes of ζ and η become negligibly small (with accuracy to the fourth significant digit). When $\rho_m = 3.5$ and 4 process converged, and when $\rho \geq 4.5$ the convergence interval decreased so much that for ζ and η found from extrapolation of curves $\zeta^{(n)}_{\rho_m}$ and $\eta^{(n)}_{\rho_m}$ (see Fig. 1) divergence of the iterative process was observed so that it was necessary to select such a $\zeta^{(0)}$ and $\eta^{(0)}$ pair for which divergence was the slowest. Figure 1 shows how stabilization of ζ and η occurred with growth of ρ_m .

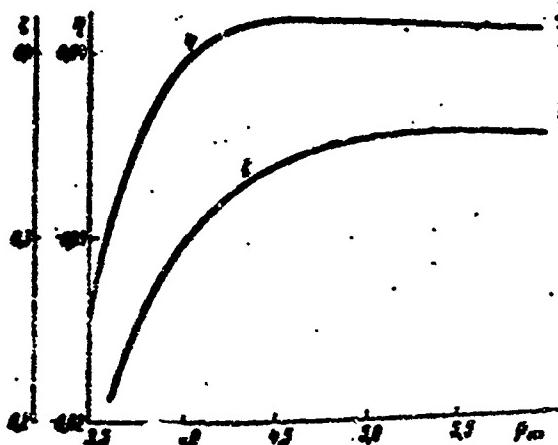


Fig. 1.

The most probable value of ζ and η are $\zeta = 0.3585$ and $\eta = -0.04154$, with accuracy to 2-3 units of the fourth digit. With growth of ρ_m the number of N of coefficients α_k necessary for finding ζ and η with an accuracy to one unit of the

fourth digit increases sharply. With $\rho_m = 5.95$ are $N = 700$ was required. Further increase of ρ_m turned out to be impossible in connection with the limited capacity of the fast store of the machine (2048 numbers) and increase of computation time. For obtaining ζ and η from values of the preceding approximation two hours of continuous machine operation were required.

From the found final values of $\zeta = 0.3585$ and $\eta = -0.04154$ a graph of $f(\rho, 0)$ was plotted, see Fig. 2. From the graph it is clear that $f(\rho, 0)$ intersects the axis of abscissas when $\rho \approx 6$. Values of $f(\rho, 0)$ for $\zeta = 0.3585$ and $\eta = -0.04157$ differ from the above-mentioned by less than on 0.001 with $\rho \leq 6$, which permits judging the accuracy of plotting of the graph. In Fig. 2 is also given a graph of function $\exp(-\alpha_1 \rho^2)$ for $\alpha_1 = 0.07170$. It is clear that up to $\rho = 5$ both curves almost coincide, so that decrease of $f(\rho, 0)$ turns out to be very rapid. The found solution is checked by direct substitution of $f(\rho, 0)$ in the form of a series in equation (108), for $\rho = 1$.

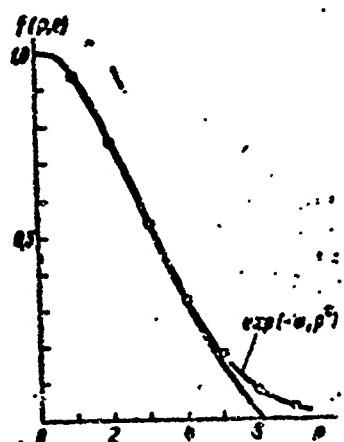


Fig. 2.

The obtained solution for $f(\rho, 0)$ is interesting basically from methodical stand point. Most interesting is the quite rapid convergence of the series representing the solution, which permits relying on analogous convergence of series giving both time-independent and time-dependent solutions. Green's function for the linear part

of the Chandrasekhar equation (25), used during the finding of $f(\rho, 0)$ can be useful during more detailed investigation of this equation.

6. Conclusions

The conducted investigation lead to the following results.

- 1) A dynamic equation was derived for determining of the scalar of correlation tensors describing two-point space-time correlation of velocities of turbulent flow and being a generalization of the Chandrasekhar equation for stationary turbulence. During derivation we adopted conditions of homogeneity and isotropy, and also the generalized hypothesis on quasi-normality of velocity distribution.
- 2) During direct derivation of the stationary equation a pair of equations not noticed by Chandrasekhar is obtained. However, these equations one should be rejected, inasmuch as it is impossible to obtain them from the more general nonstationary equation.
- 3) In principle it is possible to obtain one more equation for nonstationary turbulence, which has not yet been investigated.
- 4) The generalized Chandrasekhar equation correctly describes the last stage of degeneration of turbulence.
- 5) The Chandrasekhar equation (stationary) for the space-time longitudinal correlation function can be solved by way of the representation of solution in the form of a power series. The solution is unique if one assigns purely spatial and temporal correlation functions, which can be determined by another theory by experiment.
- 6) If a certain known combination of partial derivatives of the correlation function turns to zero at infinity, for unique solution of the Chandrasekhar equation it is sufficient to assign the spatial correlation function. Here the Euler and the temporal microscale

of turbulence turns out to be a function of the mean square of velocity, kinematic viscosity, and the integral of known form from the space correlation function.

7) The Chandrasekhar equation has a class of solutions, the form of which is universal, i.e., does not depend on the properties of liquid and intensity of turbulence. These solutions do not agree with the formula for the mean square of velocity of turbulent flow, known from the Kolmogorov theory.

8) We investigated solutions of the stationary equation not depending on time interval, and in particular, one singular solution, determined with the help of Green's function.

In conclusion we must point out that results of this work can be used during derivation and solution of equations for other space-time correlation functions in turbulent flow.

The authors thank A. I. Ivanovskiy for useful criticism and Yu. V. Sidorov for discussion of a number of questions of mathematical character.

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Submitted
31 August 1966

TURBULENCE SPECTRUM OF A STABLY STRATIFIED ATMOSPHERE

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Results of experimental investigations of the turbulence spectrum in the stratosphere and upper troposphere, where temperature stratification is stable, are examined a diagram of the generalized turbulence spectrum, including the buoyancy subdomain is offered. Particular cases of spectra with limited band of wave numbers, including the spectrum of "quasi-wave" disturbances are examined. For a physical interpretation of energy transitions in the spectrum curves $\epsilon_\theta = f(\theta)$ are used.

I. Introduction

Turbulence in the free atmosphere is the phenomenon determining in many respects the physics of processes occurring in it. Furthermore, study of atmospheric turbulence is necessary for solution of a whole series of purely applied problems of dynamics of flight and aircraft, of propagation and scattering of purities in the atmosphere, etc.

Free atmosphere is usually stratified stably in density, which noticeably affects the character of appearance and development of turbulence in it. The question of appearance of turbulence in a thermally stratified medium was solved theoretically by Richardson, who gave the criterion of growth and decrease of turbulent energy: turbulence increases when $R_i < 1$ it is not changed when $R_i = 1$, and it fades when $R_i > 1$, where

$$R_i = \frac{k}{l} \frac{1 - \frac{l_a - l}{l}}{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}. \quad (1)$$

Here from equation (1) it is clear that Richardson considered both thermal and dynamic stratification. However, for steady turbulence, R_i , i.e., the Richardson number, does not permit estimating the intensity of turbulence and, this is especially important, gives no information about scales of turbulent motion.

Since turbulence is characterized by field of random velocities, for description of the field of turbulence statistical characteristics are used. One of such characteristics, allowing us to obtain information on the intensity of turbulent motion of different scales, is the spectral density of energy distribution of turbulent motion, or the energy spectrum of turbulence

$$S(\Omega) = \frac{1}{2} \frac{\partial v_z^2}{\partial k}, \quad (2)$$

where $k = \frac{2\pi}{l}$. l – scale of motion, v_z – vorticity of corresponding scale.

For locally-isotropic uniform turbulence Kolmogorov and Obukhov, proceeding from considerations of self-similarity and assuming that energy is transmitted from larger scales to smaller without losses, obtained an expression, which in spectral form is known as the "-5/3" law

$$S(\Omega) \propto \frac{2}{3} \Omega^{-\frac{5}{3}}, \quad (3)$$

where ϵ is the rate of energy transfer over the spectra, numerically equal to the rate of dissipation of turbulent energy as heat. As can be seen from expression (3), ϵ also characterizes the intensity of turbulence.

We will assume that in turbulent atmosphere, even if we place no limitations on spatial structure and distribution of turbulent formation (i.e., also in the case of anisotropic and nonuniform turbulence), energy transfer from larger scales to smaller takes

place. In the language of spectral concepts this means that energy flux in the spectrum is directed toward greater wave numbers. Hence follows directly determination of the rate of energy transfer in a spectrum ϵ_ν as a function of wave number Ω . In particular case (3) $\epsilon_\nu = \epsilon = \text{const}$. Here, as already indicated, in a certain interval of wave numbers energy does not enter and is not expended, but is only transferred from some movements to others. In Fig. 1a in coordinates $S(\Omega)$ and Ω are depicted $\epsilon = \text{const}$ lines, where $\epsilon_1 < \epsilon_2 < \epsilon_3$. If the spectrum of turbulence is in parallel with these lines, it means that the "-5/3" law (segment BC) is valid. If, however, the spectral curve with increase of wave number passes from one ϵ to another (segment AB), it means that in a certain interval of wave numbers energy enters spectrum and, conversely, in the interval corresponding to segment CD kinetic energy of turbulent pulsations does not pass completely into energy of pulsations of smaller scales. In other words, in region A^1B^1 the source acts, but in region C^1D^1 the consumer of energy of turbulent pulsations is active.

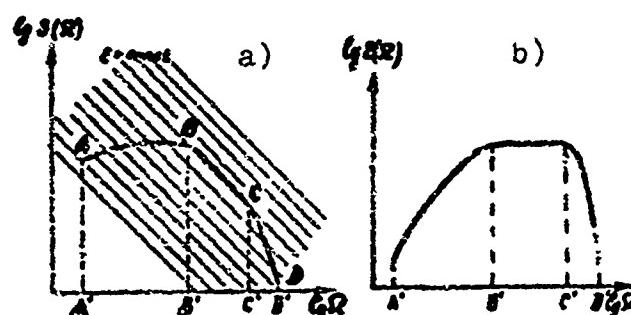


Fig. 1. Spectral density of the energy of turbulent pulsations and the rate of energy transfer over the spectrum.

A very convenient form of representation of experimental data in those cases when we want to trace transfer of energy in the turbulence spectrum is a graph of function $\epsilon_\nu = f(\Omega)$, presented in Fig. 1b. On such a graph it is possible to separate distinctly both regions of entry of energy into spectrum and regions where energy is "sucked" from the spectrum.

II. Turbulence Developed in Stably Stratified Atmosphere

Stable stratification of medium prevents the appearance in it of oscillations whatever. If, however, in such medium chaotic turbulent movements nevertheless appear, then, according to the Richardson number, this means that the dynamic factor (in the opinion of Richardson this factor is the vertical velocity gradient) predominates over thermal stability. However, the influence of Archimedian forces is not limited by the fact that they prevent the the appearance of turbulence. In well-developed turbulent flow Archimedian forces influence the character of the turbulence spectrum. The range of wave numbers (scales) in which this influence is significant is called the subdomain of buoyancy. Vortices in this region during their lifetime must accomplish work against the Archimedian forces, on which they expend part of their kinetic energy, i.e., forces of negative buoyancy are the consumer of turbulent energy here. As was indicated above, s_s drops with increase of Ω , and the spectral curve has slope greater than 5/3.

First experimental data confirming the presence in the turbulence spectrum of an interval of wave numbers in which the forces of buoyancy appear were obtained in 1959-1960 [3], [4].

Analyzing results of radio experiments, Bolgiano [4] offered a certain theoretical model of the spectrum of turbulence in the buoyancy interval. Assuming that in a defined range of wave numbers significant influence on the form of the spectrum is rendered by the rate of dissipation of mean square fluctuations of specific forces of buoyancy, Bolgiano drew a conclusion to the effect that in the interval of buoyancy the slope of the spectral curve remains constant but different from 5/3. He obtained expressions for spectra of pulsations of velocity and temperature

$$S_v(\Omega) \propto \Omega^{-\frac{11}{6}}. \quad (4)$$

$$S_r(\Omega) \propto \Omega^{-\frac{7}{6}}. \quad (5)$$

where $S_v(\Omega)$ – energy spectrum of velocities of turbulent pulsations, $S_t(\Omega)$ – energy spectrum of pulsations of temperature. During the analysis of results of experimental investigations of the spectrum of the vertical component of turbulent pulsations of velocity, which were conducted at TsAO in 1959–1960, it was found that in the interval of wave numbers corresponding to scales of from hundreds of meters to two or three kilometers the experimental curve has slope considerably greater than 5/3 (see article [3]).

Investigations were made with help of a flying laboratory in the upper troposphere basically in clear sky. Zones of intense turbulence in jet streams were inspected. Thermal stratification of the atmosphere at these heights was stable.

In work [3] an expression was offered for the turbulence spectrum, agreeing well with experimental data

$$S_v(\Omega) \propto \varepsilon_0^{\frac{2}{3}} \Omega^{-\frac{5}{3}} (1 + b\Omega^{-\frac{4}{3}}), \quad (6)$$

where $S_v(\Omega)$ – energy spectrum of vertical pulsations velocity, ε_0 – rate of dissipation of turbulent energy as heat, b – coefficient, depending on gradient of potential temperature, i.e., on the degree of thermal stability of the atmosphere.

Expression (6) was obtained from the assumption that the rate of conversion of energy in the subdomain of buoyancy depends on the wave number and is determined by the average gradient of buoyancy forces. As was shown in the cited work, expression (6) is not strict. The problem of the turbulence spectrum was more strictly theoretically solved by Lumley [6]. He started from the same considerations on dependence of the rate of transfer of energy in the spectrum on the gradient of buoyancy forces, as in article [3], and obtained the expression:

$$S_v(\Omega) = \alpha \varepsilon_0^{\frac{2}{3}} \Omega^{-\frac{5}{3}} \left[1 + \left(\frac{\Omega}{\omega_s} \right)^{-\frac{4}{3}} \right], \quad (7)$$

where Ω is wave number, characterizing the subdomain of buoyancy. Expression (7) coincides completely with expression (6). Thus it is possible to consider experimentally and theoretically established the presence in the spectrum of developed turbulence during stable stratification of the atmosphere of a subdomain of buoyancy, which is characterized by the slope of the spectral curve greater than $5/3$.

The distinction between expression (4) on the one hand and (6) and (7) on the other is reduced, as a rule, to distinction in the value of the exponent on Ω in the subdomain of buoyancy, so that in the first case the slope of the spectral curve is constant, while in the second it is a function of wave number.

In recent years in TsAO much experimental data, obtained under conditions when it was possible to expect the presence of the subdomain of buoyancy in the turbulence spectrum, has been accumulated. In Fig. 2 are presented curve of ϵ_Ω , obtained in 1965 during flights the flying laboratory in clear sky. Curves 1 and 2 were obtained for the horizontal component. Horizontal fluctuations flow velocity were measured in the region of large scales by a Doppler system [2] and in the region of small scales by an aircraft hot-wire anemometer [1].

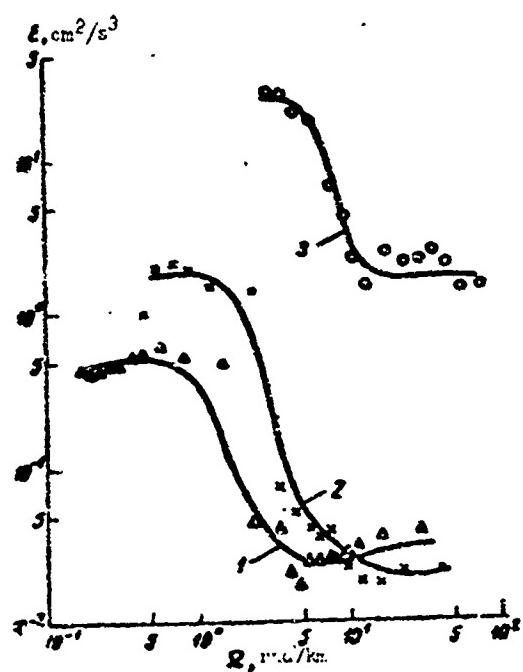


Fig. 2. Experimental curves
 $\epsilon_\Omega = f(\Omega)$: 1 and 2 - for horizontal
 and 3 - for vertical components.

Curve 3 was obtained for the vertical component according to measurements of vertical overload of the center of gravity of the aircraft, with account being taken of its transfer function. The curve lies considerably higher than 1 and 2, since it corresponds to the presence of intense bumping of the aircraft, i.e., to considerably higher energy of turbulent motion.

The curves in Fig. 2 were obtained during stable temperature stratification, and on them is seen distinctly a region of decrease of ϵ_Ω (subdomain of buoyancy). It is interesting to note that for greater intensities of turbulence this region shifts in the direction of greater wave numbers.

Experimental investigations of atmospheric turbulence widely developed during recent years made it possible to accumulate a large quantity of data on the structure of the field of turbulence. It is obvious that every method permits obtaining the spectrum of turbulence in only a defined interval of scales, but at present our concept of the spectrum of turbulence of free atmosphere from scales equal to tens of kilometers to submolecular scales, where kinetic energy of turbulent pulsations passes to kinetic energy of molecules, i.e., to heat, has become complicated. In Fig. 3 is represented schematically the energy spectrum of turbulence of stably stratified atmosphere.

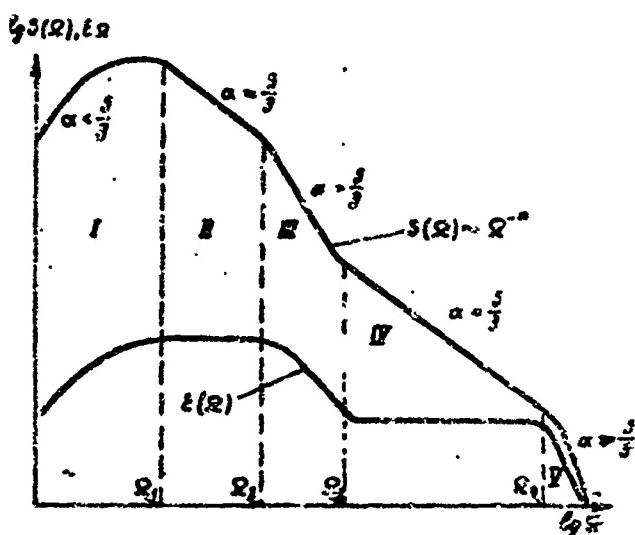


Fig. 3. Generalized curves of $S(\Omega)$ and ϵ_Ω for stably stratified atmosphere.

Region I is the region of scales in which primary turbulent formations appear. Average motion in these scales loses its stability, and part of the energy of basic flow goes to formation of disordered fluctuations. Region I is characterized by growth of ϵ_Ω . This does not mean that only this region receives turbulent energy. The increase of ϵ_Ω makes it possible to conclude only that entry of energy into the spectrum is faster than the draining off (consumption) of this energy. We will not stop here on consideration of possible mechanisms of generation of turbulent energy in region I. This question is very complex and is still far from being fully studied.

Region II is characterized by quasi-equilibrium between entry of energy into the spectrum and loss of this energy. As can be seen from the figure, the rate of energy transfer over the spectrum ϵ_Ω in this region remains almost constant.

The following region III - subdomain of buoyancy - is characterized by the fact that at wave numbers $\Omega_1 \dots \Omega_3$ acts a powerful consumer of turbulent energy while entry of energy into the spectrum from without on these scales is practically lacking. Owing to loss of energy on work against forces of buoyancy, the rate of energy transfer over the spectrum in this subdomain drops.

Region IV is the classical inertial interval. Here there is neither influx or loss of energy. The rate of transfer of energy over the spectrum in inertial interval is constant and is equal to the rate at which turbulent energy passes over the right boundary of the inertial to thermal intervals - rate of dissipation ϵ_0 .

Region V is the viscous interval, where kinetic energy, the energy of turbulence, is converted to heat. In this region the rate of energy transfer over the spectrum drops to zero.

In accordance with changes of rate of transfer of energy over the spectrum in different regions the form of spectral curve will also be changed.

From the assumption about homogeneity and local isotropy Kolmagorov and Obukhov obtained an expression for the energy spectrum of turbulence in the inertial interval - the "-5/3" spectral law

$$S(\Omega) \sim \omega^{2/3} \Omega^{-5/3}. \quad (8)$$

If by definition in the inertial interval there are no sources and consumers of turbulent energy, but only inertial transfer of energy over the spectrum from larger scales to smaller takes place, then in the interval of equilibrium (region II) both sources and consumers of turbulent energy, which, however, compensate one another, are at work. In spite of the fact that in the interval of quasi-equilibrium the condition of local isotropy is known not to be met, the spectrum in this interval also is described by the expression:

$$S(\Omega) \sim \Omega^{-5/3}. \quad (9)$$

The expression for the spectrum in the viscous interval is obtained from the Geisenberg equation [5] and has the form:

$$S(\Omega) \sim \Omega^{-7/5}. \quad (10)$$

As regards the subdomain of buoyancy, the spectral curve here has slope greater than 5/3 and is described, according to [3], by expression:

$$S(\Omega) \sim \Omega^{-5/3} (1 + \Omega^{-2/3}). \quad (11)$$

III. Discrete Spectra. Wave Disturbances in the Atmosphere

Everything said above pertains to continuous spectra. However, among experimentally obtained spectra there also are such in which turbulent motion in a wide range of scales is lacking. More precisely, this motion have amplitudes smaller than the threshold of sensitivity of the measuring equipment. Such discrete spectra are due to the presence of stable wave disturbances, not transmitting their energy to smaller scales.

Sometimes the spectrum, continuous at wave numbers $\Omega < \Omega_*$, sharply drops when $\Omega > \Omega_*$. In this case one should either recognize that the assumption about energy transfer in the continuous spectrum from larger scales to smaller is not satisfied or should assume the presence of conversion of kinetic energy pulsations when $\Omega > \Omega_*$ to some other form of energy.

We will, as before, consider that in turbulent flow energy transfer from larger scales to smaller takes place. As already stated, growth of the rate of energy transfer ϵ_Ω indicates the presence of a source of turbulent energy and, conversely, a decrease of ϵ_Ω indicates the presence of a consumer.

When spectrum of turbulence, and consequently and $\epsilon_\Omega = f(\Omega)$, are continuous, such description, in general, is trivial. However, quite another picture is obtained if with the same assumption about the direction of energy flow in the spectrum we turn to consideration of stable wave disturbances in a real medium, i.e., in a medium possessing dissipative properties (for example, viscosity).

Let us examine forced oscillations of air with frequency corresponding to wave number Ω . We know that if the source of these oscillations ceases to act, the oscillations will fade with time owing to internal friction (viscosity) of air. Let us try to analyze the meaning of this.

An air mass has defined viscosity, depending on temperature and density. Molecules of air are in random motion, where the number of molecules in a unit volume and their average velocity determine the viscosity of the air.

When we say that oscillation of air fade, it means that the mechanical energy of these oscillations turns to energy of molecular motion.

If continuous wave oscillations exist in the atmosphere, it means that they are supported constantly by an active external source.

The scale of disturbances generated by this source should, in any case, be no less than the length of the continuous wave. Consequently, in case of stable wave disturbances also the spectral energy flow preserves its direction toward larger wave numbers, and energy going to larger scales is transferred to the region of submolecular scales. The spectrum in this case is discrete and has the form presented in Fig. 4.



Fig. 4. Spectrum of purely wave disturbance in viscous medium.

On the axis of abscissas to the first peak corresponds the wavelength of stable oscillation; to the second corresponds the value of the order of the mean free path of molecules.

Thus if the medium possesses viscosity, "sucking out" of energy of any mechanical motion appearing in this medium will occur. It is possible to apply the same reasoning to a medium in which random motion with scales much larger than molecular exists, that is, to a medium possessing turbulent viscosity. If in an air mass there already are movements of defined scales, the intensity of these movements can increase directly as a result of energy of other mechanical motion, even if its scale is much greater. From this stand point it is easy to explain the character of the spectral curve obtained by Van der Hoven [7], having dips over a wide range of wave numbers.

Let us now consider cases when the spectrum is continuous over a rather wide range of wave numbers, but then drops sharply.

In examining of such rapidly dropping spectra we will, as earlier, consider that spectral energy flow is directed toward larger wave numbers. Inasmuch as such spectra are characteristic for stably

stratified atmosphere, one should assume that rapid drop of spectrum is explained by the influence of forces of buoyancy, which consume almost all the energy arriving from the direction of smaller wave numbers.

IV. Turbulence in a Thermally Stable Stratified Medium

With strong thermal stability of atmosphere which for example, is typical of the stratosphere, for disturbance of laminarity flow large wind gradients are necessary. The very fact of appearance and development of turbulence in such a medium indicates that this condition (1) is fulfilled.

However, although the reserve of thermal stability is insufficient to impede generation of turbulence, the forces of buoyancy intensively counteract cascade energy transfer over the spectrum. Three cases, corresponding to the three curves in Fig. 5 are possible here.

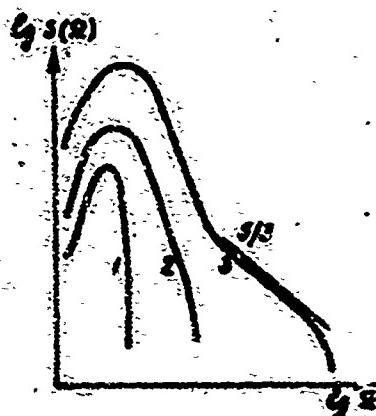


Fig. 5. Modifications of turbulence spectra for different relationships between intensities of source and consumer of turbulent energy and for their different characteristic scales.

1. With very strong thermal stability of medium the spectrum of turbulence can be localized in a narrow band of wave numbers. In this case the spectrum obtained as a result of treatment of realization of finite length for practical purposes will not differ from the spectrum of purely harmonic oscillation, if we use existing methods of statistical computer treatment. Such "quasi-wave" disturbances

differ from those of "pure wave" type by the fact that in them energy transfer from larger scales to smaller takes place, and then all energy is expended on work against the forces of buoyancy.

2. With somewhat lesser thermal stability of atmosphere, as well as with its greater dynamic instability, and also in those cases when the source of turbulent energy is located in the region of wave numbers distant from the subdomain of buoyancy, the spectrum of turbulence is found considerably broader and differs significantly from the spectrum of purely harmonic oscillations. However, even in this case the forces of buoyancy absorb all the kinetic energy of turbulent pulsations.

3. The most frequently encountered case is that when there is a continuous spectrum of turbulence, in which kinetic energy is transferred from larger scale to smaller. Part of the energy becomes potential energy, and then in scales where the forces of buoyancy no longer play an essential role there is inertial transfer in accordance with the "-5/3" law. In the end all energy turns to heat in the viscous interval.

Diverse variants of spectra, shown in Fig. 5, are particular cases of the general diagram, presented in Fig. 3, and characterize distribution of energy in the spectrum of developed turbulence, i.e., the stationary case. During experimental investigations of turbulence it sometimes is possible to obtain spectra of developing or fading turbulence. Such spectra have complicated form, and on them are distinctly seen local (according to wave numbers) sources of turbulent energy, such, for example, as disintegrating gravitational waves.

For the spectrum of developed turbulence in a stably stratified medium it is characteristic that the source of turbulent energy lies in the region of small wave numbers, while the spectral curve itself has only one peak in the region of generation.

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Submitted
26 September 1966